MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 4

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SUMMER 2019

- **Submit questions 2 and 5 for grading.**
- **Due on: Wednesday June 5, 2019.**
- **You will have a quiz from the following questions on Wednesday June 5, 2019.**

(1) Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}[x] \to \mathbb{R}^2$ by
$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^{2} x^3 p(x)\ dx + c\sin\big(p(0)\big)\right).$$
Show that T is linear if and only if $b = c = 0$.

(2) Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $(Tv_1, \ldots, Tv_m)$ is linearly independent list of vectors in W. Prove that $v_1, v_2, \ldots, v_m$ are linearly independent.

(3) Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that
$$f(\alpha v) = \alpha f(v)$$
for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^2$, but f is not linear.

(4) Check that the function $f : \mathbb{C} \to \mathbb{C}$ defined as
$$f(z) = \text{Re } z$$
satisfies
$$f(u + v) = f(u) + f(v)$$
for all $u, v \in \mathbb{C}$, but f is not linear.

(5) Suppose V is a vector space and $S, T \in \mathcal{L}(V)$ are such that
$$\text{range } S \subseteq \text{null } T.$$
Prove that $(ST)^2 = 0$.

(6) Show that
$$U := \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4);\ \dim\ \text{null } T > 2\}$$
is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

(7) Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that
$$\text{range } T = \text{null } T.$$

(8) Prove that there does not exist a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that
$$\text{range } T = \text{null } T.$$

(9) Suppose that V is finite dimensional vector space and let $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \text{null } T = \{0\}$ and $\text{range } T = \{Tu;\ u \in U\}$.