Summary: Existence and Uniqueness of solutions

The Rank Theorem

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix.

- The system may not be consistent, in which case it has no solution.
- To decide whether the system is consistent, check that $\mathbf{b}$ is in the column space of A.
- If the system is consistent, then
  - Either $\text{rank}(A) = n$ (which also means that $\dim \text{null}(A) = 0$), and the system has a unique solution.
  - Or $\text{rank}(A) < n$ (which also means that $\text{null}(A)$ is non-trivial), and the system has an infinite number of solutions.

Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is an $m \times n$ matrix.

- If $m = n$ and the system is consistent, then
  - Either $\det(A) \neq 0$, in which case rank$(A) = n$, dim null(A)=0, and the system has a unique solution.
  - Or $\det(A) = 0$, in which case dim null$(A) > 0$, rank$(A) < n$, and the system has an infinite number of solutions.
- Note that when $m = n$, having $\det(A) = 0$ means that the columns of A are linearly dependent.
- It also mean that null(A) is non-trivial and that rank$(A) < n$.