Test 2 **Linear Algebra MA 413** **April 10, 2017**

Name: Key **Signature:** ____________________

SHOW ALL YOUR WORK!

In all of the following questions, V is a finite dimensional vector space over a field $\mathbb{F}$.

1. [20 points] Show that the linear map $T : \mathbb{F}^4 \to \mathbb{F}^2$ is surjective if

$$\text{null}(T) = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 | x_1 = 3x_2, x_3 = x_4\}$$

$\text{null}(T) = \{ (3x_2, x_2, x_3, x_3) \in \mathbb{F}^4 \}$

Since

$(3x_2, x_2, x_3, x_3) = x_2(3, 1, 0, 0) + x_3(0, 0, 1, 1)$

and $(3,1,0,0)$ & $(0,0,1,1)$

are linearly indep.

$\Rightarrow \dim(\text{null}(T)) = 2.$

dim. formula gives

$\dim \mathbb{F}^4 = \dim(\text{null}(T)) + \dim(\text{range}(T))$

$4 = 2 + \dim(\text{range}(T))$

$\Rightarrow \dim(\text{range}(T)) = 2 = \dim(\mathbb{F}^2)$

$\Rightarrow \text{range}(T) = \mathbb{F}^2$

$\Rightarrow T$ is surjective.

2. [15 points] Suppose that U and W are subspaces of V. Prove that if

$$\dim(V) < \dim(U) + \dim(W) \qquad (*)$$

then $U \cap W \neq \{0\}$.

$U + W$ is a subspace of V

$$\dim(U+W) \leq \dim V \qquad (1)$$

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$

$$\dim V \underset{\text{using (1)}}{\geq} \dim(U+W) \underset{\text{using } (*)}{>} \dim V - \dim(U \cap W)$$

$$\Rightarrow \quad \dim(U \cap W) > 0$$

$$\Rightarrow \quad U \cap W \neq \{0\}.$$

)

3. [20 points] Let $V := \mathbb{C}_n[z]$ is the vector space over $\mathbb{C}$ consisting of all polynomials with degree less than or equal to n. Let $T \in \mathcal{L}(V)$ be the differentiation map

$$T(q) = q'.$$

① • Give a basis for V then find $M(T)$, i.e., the matrix representation of T, under that basis.

② • Find all the eigenvalues of T.

① $B = (1, z, z^2, \ldots, z^n)$

$T1 = 0$

$Tz = 1$

$Tz^2 = 2z$

$Tz^3 = 3z^2$

$\vdots$

$Tz^n = nz^{n-1}$

$$M(T) = \begin{array}{c|ccccc} & T1 & Tz & Tz^2 & \cdots & Tz^n \\ \hline 1 & 0 & 1 & 0 & \cdots & 0 \\ z & 0 & 0 & 2 & \cdots & 0 \\ z^2 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ z^{n-1} & 0 & 0 & 0 & & n \\ z^n & 0 & 0 & 0 & & 0 \end{array}$$

$(n+1)(n+1)$

② zero is the only evalue of T

because $M(T)$ is upper triangular with zeros on the diagonal.

4. [20 points] Let V and W be vector spaces over $\mathbb{F}$, and suppose that $T \in \mathcal{L}(V, W)$ is injective. Given a linearly independent list of vectors $(v_1, \ldots, v_n)$ of V, prove that the list $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

let $a_1, a_2, \ldots, a_n \in \mathbb{F}$ such that

$$a_1 Tv_1 + \cdots + a_n Tv_n = 0$$

$$\Rightarrow T(a_1 v_1 + a_2 v_2 + \cdots + a_n v_n) = 0$$

since T is injective

$$\Rightarrow a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$$

because $(v_1, \ldots, v_n)$ are linearly independent.

5. [20 points] Let $T \in \mathcal{L}(V)$ be a linear map, and suppose that λ is an eigenvalue of T. Let $U = \operatorname{range}(T - \lambda \mathbb{1})$.

(a) Show that $\dim U < \dim V$.

(b) Prove that U is T-invariant.

$*$ U is a subspace of V.

$\boxed{a}$ using the dimension formula,

$$\dim(V) = \dim(\operatorname{null}(T-\lambda\mathbb{1})) + \dim(\operatorname{range}(T-\lambda\mathbb{1})) \quad \text{---} (*)$$

since λ is an e-value of T

$$\Rightarrow \operatorname{null}(T-\lambda\mathbb{1}) \neq \{0\}$$

$$\Rightarrow \dim(\operatorname{null}(T-\lambda\mathbb{1})) > 0$$

$$\Rightarrow \dim(\operatorname{range}(T-\lambda\mathbb{1})) < \dim(V) \text{ ---}$$

from $(*)$

$\boxed{b}$ let $u \in U$

$$Tu = \underbrace{(T-\lambda\mathbb{1})u}_{\in \operatorname{range}(T-\lambda\mathbb{1}) = U} + \underbrace{\lambda u}_{\in U} \in U$$

6. [15 points] Suppose $T \in \mathcal{L}(V)$ and the set $B = \{v_1, v_2, \cdots, v_n\}$ is a basis of V for which $M(T)$, the matrix corresponding to T in the basis B, is upper triangular. Prove that if T is invertible, then the matrix $M(T^{-1})$, the matrix corresponding to T^{-1} in the basis B, is also upper triangular.

$M(T)$ is upper triangular & invertible $\Rightarrow$ all diagonal entries are non zeros.

suppose that $M(T) = (a_{ij})_{1 \le i,j \le n}$ with $a_{ij} = 0$ for $i > j$ $(a_{jj} \ne 0 \ \forall j)$

we need to show that $T^{-1} v_j \in \text{span}(v_1, \ldots, v_j)$ for all $j = 1, 2, \ldots, n$ — $(*)$

by induction:

for $j = 1$ $\quad T v_1 = a_{11} v_1 \iff v_1 = a_{11} T^{-1} v_1$

$\iff T^{-1} v_1 = \frac{1}{a_{11}} v_1$

$\Rightarrow T^{-1} v_1 \in \text{span}(v_1)$

suppose $(*)$ is correct for all $j = 1, 2, \ldots, k-1$, want to show that it is correct also for $j = k$.

$$T v_k = a_{1k} v_1 + a_{2k} v_2 + \ldots + a_{(k-1)k} v_{k-1} + a_{kk} v_k$$

$$\Rightarrow \frac{1}{a_{kk}} T v_k = \frac{a_{1k}}{a_{kk}} v_1 + \frac{a_{2k}}{a_{kk}} v_2 + \cdots + \frac{a_{(k-1)k}}{a_{kk}} v_{k-1} + v_k$$

$$\Rightarrow \frac{1}{a_{kk}} v_k = \frac{a_{1k}}{a_{kk}} T^{-1} v_1 + \ldots + \frac{a_{(k-1)k}}{a_{kk}} T^{-1} v_{k-1} + T^{-1} v_k$$

$$\Rightarrow T^{-1} v_k = \frac{-a_{1k}}{a_{kk}} \underbrace{T^{-1} v_1}_{\in\, \text{span}(v_1)} + \ldots + \frac{-a_{(k-1)k}}{a_{kk}} \underbrace{T^{-1} v_{k-1}}_{\in\, \text{span}(v_1, \ldots, v_{k-1})} + \frac{1}{a_{kk}} v_k$$

$$\Rightarrow T^{-1} v_k \in \text{span}(v_1, \ldots, v_{k-1}, v_k)$$