Test 1 **Linear Algebra MA 413** **February 22, 2017**

Name: ______________________ **Signature:** ______________________

SHOW ALL YOUR WORK!

In all of the following questions, V is a finite dimensional vector space over a field $\mathbb{F}$.

1. [15 points] Solve the equation $z^3 - 4i = 0$, where $z \in \mathbb{C}$.

2. [10 points] Determine whether the following set of vectors are linearly independent in $\mathbb{R}^3$.

$$v_1 = (1, 1, 1), \quad v_2 = (0, 1, -1), \quad v_3 = (1, 1, 0)$$

3. [15 points]

   (a) List all subspaces of $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

   (b) Give an example of a nonempty subset of $\mathbb{R}^2$ that is closed under scalar multiplication but it is not a subspace of $\mathbb{R}^2$.

   (c) Give an example of a nonempty subset of $\mathbb{R}^2$ that is closed under vector addition but it is not a subspace of $\mathbb{R}^2$.

4. [20 points] Let V_1 and V_2 be two subspaces of V,

   (a) Prove that $V_1 \cap V_2$ is a subspace of V.

   (b) Prove that $V_1 + V_2$ is a subspace of V.

   (c) Give examples of V, and subspaces V_1 and V_2 of V in each of the following cases

      i. $V_1 \cup V_2$ is a subspace of V.

      ii. $V_1 \cup V_2$ is not a subspace of V.

5. [10 points] Let $v_j \in V$ for $j = 1, 2, \ldots, n$. Write a mathematical definition of each of the following

   (a) The list of vectors $(v_1, v_2, \ldots, v_n)$ is linearly dependent.

   (b) $V = \text{span}(v_1, v_2, \ldots, v_n)$.

6. [20 points] Let W_1 and W_2 be two subspaces of V, and suppose that $V = W_1 + W_2$, prove that $V = W_1 \oplus W_2$ if and only if $W_1 \cap W_2 = \{0\}$.

7. [20 points] Suppose that $(v_1, v_2, \ldots, v_n)$ be a linearly independent list of vectors in V. Given any $w \in V$ such that

$$(v_1 + w, v_2 + w, \ldots, v_n + w)$$

is a linearly dependent list of vectors in V, prove that $w \in \operatorname{span}(v_1, v_2, \ldots, v_n)$.

8. [10 points] Suppose that W_1 and W_2 are two subspaces of V such that $W_1 \cap W_2 = \{0\}$. Let $(v_1, v_2, \ldots, v_m)$ and $(w_1, w_2, \ldots, w_n)$ be two linearly independent sets in the subspaces W_1 and W_2, respectively. Prove that

$$(v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m)$$

is linearly independent in $W_1 \oplus W_2$.