HW #3

Galen

P46 Q1 (f) $S := \left\{ \begin{bmatrix} a & a+b \\ a+b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$ under $+$ & $\cdot$ of $\mathbb{R}^{2\times 2}$

it is easy to show that $\mathbb{R}^{2\times2}$ is a vector space under the addition

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$$

and

scalar multiplication $\alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$, $\alpha \in \mathbb{R}$.

so it is enough to show that S is a subspace of $\mathbb{R}^{2\times2}$.

① $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$

② let $A, B \in S$

i.e,

$A = \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix}$ for some $a, b \in \mathbb{R}$

and $B = \begin{bmatrix} c & c+d \\ c+d & c \end{bmatrix}$ for some $c, d \in \mathbb{R}$

$$A + B = \begin{bmatrix} a+c & a+b+c+d \\ a+b+c+d & b+c \end{bmatrix} = \begin{bmatrix} a+c & (a+c)+(b+c) \\ (a+c)+(b+d) & b+c \end{bmatrix} \in S$$

③ $\alpha A = \begin{bmatrix} \alpha a & \alpha(a+b) \\ \alpha(a+b) & \alpha b \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha a+\alpha b \\ \alpha a+\alpha b & \alpha b \end{bmatrix} \in S$

$\alpha \in \mathbb{R}$

Q1 (g) p46

$$\left\{ \begin{bmatrix} a & a+b+1 \\ a+b & a \end{bmatrix}, a, b \in \mathbb{R} \right\}$$

is not a vector space because it does not contain the addition identity.

Calcu. p46

Q3 [a] $S_1 = \{ f \in \mathcal{C}(\mathbb{R}) \mid f(x) \le 0, \ \forall x \in \mathbb{R} \}$

is not a subspace because it is not closed under the scalar multiplication for example

if $f \in S_1$ with $f \neq 0$ then $(-2)f \notin S_1$

[b] $S_2 = \{ f \in \mathcal{C}(\mathbb{R}) \mid f(0) = 0 \}$ is a subspace of $\mathcal{C}(\mathbb{R})$

1. $f(x) = 0 \ \forall x \in \mathbb{R}$ is a vector in S_2
2. let $f_1, f_2 \in S_2$, means that $f_1(0) = f_2(0) = 0$

   then

   $(f_1 + f_2) \in \mathcal{C}(\mathbb{R})$ and $(f_1 + f_2)(0) = f_1(0) + f_2(0)$

   $= 0 + 0$

   $= 0$

   $\Rightarrow f_1 + f_2 \in S_2$
3. let $f \in S_2$, $\alpha \in \mathbb{R}$

   then $(\alpha f)(0) = \alpha f(0) = 0$

   $\Rightarrow \alpha f \in S_2$.

Calm.

Q3 p46 $\boxed{c}$ $S_3 = \{ f \in \mathcal{C}(\mathbb{R}),\ f(0) = 2 \}$

is not a subspace of $\mathcal{C}(\mathbb{R})$ because
$f(x) = 0$ is not in S_3.

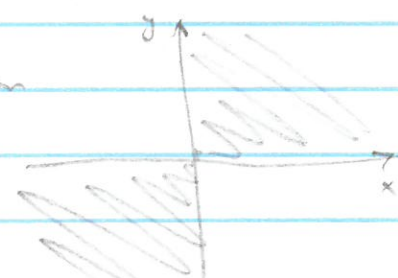
$\boxed{d}$ the set of all constant functions
is a subspace of $\mathcal{C}(\mathbb{R})$
because constant functions $\in \mathcal{C}(\mathbb{R})$ and

1. $f(x) = 0$ is a constant function.
2. the sum of two constant functions is a constant function.
3. for any $\alpha \in \mathbb{R}$, α times a constant function is a constant function.

Calcula. Q4 p47

let $U = \{ (x, y) \in \mathbb{R}^2 ;\ x \geq 0 \wedge y \leq 0 \text{ OR } x \leq 0 \wedge y \geq 0 \}$

(the set of all points in the first and 3^{rd} Quadrants)



U is closed under the scalar multiplication

let $\alpha \in \mathbb{R}$ then

$\alpha(x, y) = (\alpha x, \alpha y)$

note that
- if $\alpha > 0$ then $(\alpha x, \alpha y)$ is a point in the same quadrant as (x, y)
- if $\alpha < 0$ then $(\alpha x, \alpha y)$ is a point in the opposite quadrant of (x, y)
- if $\alpha = 0$ then $(\alpha x, \alpha y) = (0, 0) \in U$.

and hence, U is closed under scalar multiplication.

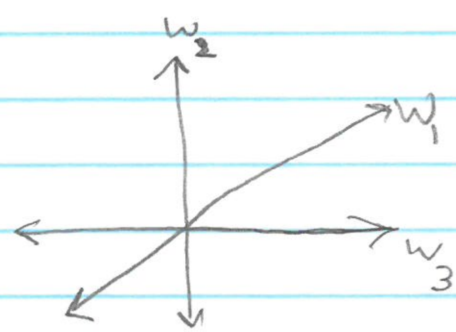
but it is not closed under vector addition, for example

$$\underset{\substack{\cap\\ U}}{(-1,5)} + \underset{\substack{\cap\\ U}}{(0,-1)} = (-1,4) \notin U.$$

Q3 Proof-Writing Ex.

The claim is not correct for example, let V be any nontrivial vector space (i.e., $V \neq \{0\}$)

$$V+V = \{0\}+V \quad \text{but} \quad V \neq \{0\}.$$



Q4 Proof-writing Ex.

The claim is not correct

Counter example: let $W_3 = \{(x,0) \in \mathbb{R}^2\}$ *
$W_2 = \{(0,y) \in \mathbb{R}^2\}$
$W_1 = \{(x,x) \in \mathbb{R}^2\}$ see the picture

W_1, W_2, W_3 are subspace of $\mathbb{R}^2$ (lines).

Note that $\mathbb{R}^2 = \underbrace{W_1 \oplus W_3}_{W_1 \cap W_3 = \{(0,0)\}} = \underbrace{W_2 \oplus W_3}_{W_2 \cap W_3 = \{(0,0)\}}$, but $W_1 \neq W_3$.