Chapter 1 Calculational Ex:

p9 **1 (b)**

$$x + 2y - 3z = 4 \quad \text{--- (1)}$$
$$x + 3y + z = 11 \quad \text{--- (2)}$$
$$2x + 5y - 4z = 13 \quad \text{--- (3)}$$
$$2x + 6y + 2z = 22 \quad \text{--- (4)}$$

Note that (2) x 2 gives (4) thus the system reduced to three equations with three variables.

$$(4) + (-1)(3) \rightarrow y + 6z = 9 \quad \text{--- (5)}$$
$$(-2)(1) + (3) \rightarrow y + 2z = 5 \quad \text{--- (6)}$$

two eq. with two variables

$$(5) + (-1)(6) \rightarrow 4z = 4 \Rightarrow \boxed{z = 1}$$

subs. $z = 1$ in (6) $\rightarrow y + 2 = 5 \Rightarrow \boxed{y = 3}$

subs. $z = 1$ & $y = 3$ in (1) $\rightarrow x + 2(3) - 3(1) = 4$

$$\boxed{x = 1}$$

the solution set $\{(1,3,1)\}$.

- The system can be represented as follows

$$f : R^3 \rightarrow R^4$$

$$f(x,y,z) = (x + 2y - 3z,\ x + 3y + z,\ 2x + 5y - 4z,\ 2x + 6y - 2z)$$

the system is the equation $f(x,y,z) = (4, 11, 13, 22)$.

P9 $\boxed{\text{Q1 c}}$

$$x + 2y - 3z = -1 \quad \text{—— (1)}$$
$$3x - y + 2z = 7 \quad \text{—— (2)}$$
$$5x + 3y - 4z = 2 \quad \text{—— (3)}$$

$2 \cdot (2) + (1) \longrightarrow 7x + z = 13$ —— (4)

$3 \cdot (2) + (3) \longrightarrow 14x + 2z = 23$ —— (5)

$(-2) \cdot (4) + (5) \longrightarrow 0 = -3$!!

the system has no solution.

- $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$f(x, y, z) = (x + 2y - 3z,\ 3x - y + 2z,\ 5x + 3y - 4z)$.

the system can be written as

$$f(x, y, z) = (-1, 7, 2).$$

p 10

CH1 Proof-writing Ex:

$$ax_1 + bx_2 = 0 \quad — (1)$$
$$cx_1 + dx_2 = 0 \quad — (2)$$

Prove that if $ad - bc \neq 0 \Longrightarrow x_1 = x_2 = 0$ is the only solution.

first note that $x_1 = x_2 = 0$ is a solution of the system.

$(-c) \cdot (1) + (a) \cdot (2) \rightarrow -cbx_2 + adx_2 = 0$

$\Rightarrow (ad - cb)\, x_2 = 0$

$\Rightarrow x_2 = 0$ (because $ad - cb \neq 0$)

by subs. $x_2 = 0$ in (1) and (2)

$\Rightarrow ax_1 = 0$ and $cx_1 = 0$

Note that if $a = c = 0$ then $ad - bc = 0 - 0 = 0$!!

So, without loss of generalization, we may assume that $a \neq 0$

and since $ax_1 = 0$ then $\boxed{x_1 = 0}$

Thm 2.3.5 p 23 (part 1)

$z_1, z_2 \in \mathbb{C}$

1. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ and $e^{z} \neq 0$ for any choice of $z \in \mathbb{C}$.

<u>proof</u> we start with the first statement $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$

RHS: $e^{z_1} \cdot e^{z_2} = e^{x_1+iy_1} \cdot e^{x_2+iy_2} = e^{x_1} \cdot e^{iy_1} \cdot e^{x_2} \cdot e^{iy_2}$

$= e^{x_1+x_2} \cdot e^{i(y_1+y_2)}$ using Lemma 2.3.2 with $r_1 = e^{x_1}$, $r_2 = e^{x_2}$

$= e^{(x_1+x_2)+i(y_1+y_2)}$ def. of the exponential function

$= e^{(x_1+iy_1)+(x_2+iy_2)} = e^{z_1+z_2} = LHS$

<u>Method 2</u>: by rewriting $e^{iy_1} = \cos y_1 + i \sin y_1$

$\vdots$

for the second part, Assume that $e^{z} = 0$ (for $z = x+iy$)

$\Rightarrow e^{x} \cdot e^{iy} = 0 \Rightarrow e^{x}(\cos y + i \sin y) = 0$

$\Rightarrow \cos y + i \sin y = 0$ (because $e^{x} \neq 0$)

real valued

$\Rightarrow \cos y = 0$ and $\sin y = 0$

(contradiction).

thus, $e^{z} \neq 0$ for any $z \in \mathbb{C}$

Chapter 2:

Calculational Exercises:

p24 * $\boxed{1\ c}$ $\frac{2+3i}{4+i} = \frac{2+3i}{4+i} \cdot \frac{4-i}{4+i} = \frac{(8+3)+i(-2+12)}{16+1}$

$= \frac{1}{17}(11+10i) = \frac{11}{17} + \frac{10}{17}i$

p24 * $\boxed{1\ d}$ $\frac{1}{i} + \frac{3}{1+i}$

$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$ —— 1

$\frac{3}{1+i} = \frac{3}{1+i} \cdot \frac{1-i}{1-i} = \frac{3-3i}{2} = \frac{3}{2} - \frac{3}{2}i$ 2

$1 + 2 = -i + \frac{3}{2} - \frac{3}{2}i = \frac{3}{2} + \left(-1-\frac{3}{2}\right)i = \frac{3}{2} - \frac{5}{2}i$

p24 * $\boxed{1\ e}$ $(-i)^{-1} = \frac{1}{-i} \cdot \frac{i}{i} = \frac{i}{1} = i$

p24 * $\boxed{1\ f}$ $(-1+i\sqrt{3})^3 = (-1+i\sqrt{3})(-1+i\sqrt{3})(-1+i\sqrt{3})$

$= \left((1-3)+i(-\sqrt{3}-\sqrt{3})\right)(-1+i\sqrt{3})$

$= (-2-2\sqrt{3}i)(-1+i\sqrt{3})$

$= (2+6) + \left(2\sqrt{3} + (-2\sqrt{3})\right)i$

$= \boxed{8}$

$z = x + iy$

p24 $\boxed{\text{Q2 c}}$

$$\frac{z+1}{2z-5} \cdot \frac{2\bar{z}-5}{2\bar{z}-5} = \frac{2z\bar{z} - 5z + 2\bar{z} - 5}{|2z-5|^2}$$

$$4|z|^2 - 10z - 10\bar{z} + 25 = 4|z|^2 - 10(z+\bar{z}) + 25 = 4(x^2+y^2) - 10(2x) + 25 = 4x^2 + 4y^2 - 20x + 25$$

$$= \frac{2|z|^2 - 5z + 2\bar{z} - 5}{4x^2 + 4y^2 - 20x + 25}$$

$$= \frac{2(x^2+y^2) - 5(x+iy) + 2(x-iy) - 5}{4(x^2+y^2) - 20x + 25}$$

$$= \frac{2(x^2+y^2) - 5x + 2x - 5 + i(-5y - 2y)}{4(x^2+y^2) - 20x + 25}$$

$$= \frac{2(x^2+y^2) - 3x - 5}{4(x^2+y^2) - 20x + 25} + \left(\frac{-7y}{4(x^2+y^2) - 20x + 25}\right) i$$

p24 $\boxed{\text{Q2 d}}$

$$z^3 = (x+iy)(x+iy)(x+iy)$$

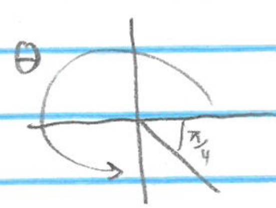
$$= \left[(x^2-y^2) + 2ixy\right](x+iy)$$

$$= \left((x^2-y^2)x - (2xy\,y)\right) + \left((x^2-y^2)y + 2xy \cdot x\right) i$$

$$= (x^3 - 3xy^2) + (3x^2y - y^3) i$$

p24 $\boxed{\text{Q3}}$

$$z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$$

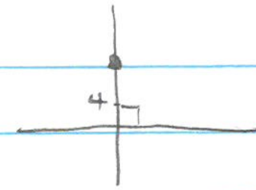


$$r = |z| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\theta = 2\pi - \pi/4 = \frac{7\pi}{4} \Rightarrow \boxed{z = e^{\frac{7\pi}{4} i}}$$

p24 $\boxed{Q4d}$

$$z^3 = 4i$$



$$\Rightarrow \quad z^3 = 4e^{\frac{\pi}{2}i}$$

$\Rightarrow$ the roots are z_0, z_1, z_2 where

$$z_0 = \sqrt[3]{4}\; e^{\frac{i}{3}\frac{\pi}{2}} = \sqrt[3]{4}\; e^{\frac{\pi}{6}i}$$

$$z_1 = \sqrt[3]{4}\; e^{\frac{i}{3}\left(\frac{\pi}{2}+2\pi\right)} = \sqrt[3]{4}\; e^{\frac{5\pi}{6}i}$$

$$z_2 = \sqrt[3]{4}\; e^{\frac{i}{3}\left(\frac{\pi}{2}+4\pi\right)} = \sqrt[3]{4}\; e^{\frac{9\pi}{6}i} = \sqrt[3]{4}\; e^{\frac{3}{2}\pi}$$

p25 $\boxed{Q5b}$

$$\overline{\left(\frac{(8-2i)^{10}}{(4+6i)^5}\right)} = \frac{\overline{(8-2i)^{10}}}{\overline{(4+6i)^5}} = \frac{\left(\overline{8-2i}\right)^{10}}{\left(\overline{4+6i}\right)^5} = \frac{(8+2i)^{10}}{(4-6i)^5}.$$

p25 $\boxed{Q7}$

$$e^{e^z} = e^{e^{x+iy}} = e^{e^x(\cos y + i\sin y)} = e^{e^x\cos y} \cdot e^{(e^x\sin y)i}$$

$$= e^{e^x\cos y}\left[\cos\left(e^x\sin y\right) + i\sin\left(e^x\sin y\right)\right]$$

$$= \underbrace{e^{e^x\cos y}\cos\left(e^x\sin y\right)}_{\text{Real part}} + i\,\underbrace{e^{e^x\cos y}\sin\left(e^x\sin y\right)}_{\text{Im. part}}$$

CH2 Proof-Writing Ex

p25 ① $a \in \mathbb{R}$, $z, w \in \mathbb{C}$. let $z = \cancel{x}+iy$, $x, y \in \mathbb{R}$

$\boxed{a}$ • $Re(az) = Re(a(x+iy)) = Re(ax + iay)$

$= ax$

$= a\,Re(z)$.

• $Im(az) = Im(a(x+iy)) = Im(ax + iay)$

$= ay$

$= a\,Im(z)$

p25 $\boxed{b}$ let $w = w_1 + iw_2$

• $Re(z+w) = Re((x+iy)+(w_1+iw_2)) = Re(x+w_1 + i(y+w_2))$

$= x+w_1 = Re(z) + Re(w)$

• $Im(z+w) = Im(x+w_1 + i(y+w_2))$

$= y + w_2 = Im(z) + Im(w)$.

p25 $\boxed{\text{Q3 p 25}}$

$z, w \in \mathbb{C}$

we need to prove that

$$|z-w|^2 + |z+w|^2 = 2\left(|z|^2 + |w|^2\right)$$

first, note that

$$|z-w|^2 = (z-w)\overline{(z-w)} = (z-w)(\bar{z} - \bar{w})$$

$$= |z|^2 - w\bar{z} - z\bar{w} + |w|^2.$$

$$|z+w|^2 = (z+w)(\bar{z} + \bar{w})$$

$$= |z|^2 + w\bar{z} + z\bar{w} + |w|^2$$

thus the LHS $= |z|^2 - \cancel{w\bar{z}} - \cancel{z\bar{w}} + |w|^2 + |z|^2 + \cancel{w\bar{z}} + \cancel{z\bar{w}} + |w|^2$

$$= 2|z|^2 + 2|w|^2 = \text{RHS}.$$