Test 2 | **Linear Algebra** | **June 24, 2020**

MATH 413/513 | **Dr. Abdul-Rahman**

Name: ______________________ **Signature:** ______________________

Show all your work!

In the following questions: U, V, and W are finite dimensional vectors spaces over a field $\mathbb{F}$.

(1) [20 points] Prove that the map $S : \mathcal{P}_4(\mathbb{F}) \to \mathbb{F}^{5,1}$ defined by

$$Sp = \mathcal{M}(p)$$

is an isomorphism of $\mathcal{P}_4(\mathbb{F})$ onto $\mathbb{F}^{5,1}$, where $\mathcal{M}(p)$ is the matrix of $p \in \mathcal{P}_4(\mathbb{F})$ with respect to the standard basis $(1, z, z^2, z^3, z^4)$.

(2) [20 points] Let $S \in \mathcal{L}(U, V)$.
  (a) Show that there exists a subspace W of U such that $U = \text{null}(S) \oplus W$.
  (b) Suppose that $(w_1, \ldots, w_k)$ is a basis of W in part (a). Show that $(Sw_1, \ldots, Sw_k)$ is a basis for range(S).

(3) [10 points] Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Prove that

$$\dim\big(\text{null}(S)\big) \leq \dim\big(\text{null}(T \circ S)\big).$$

(4) [10 points] Prove that the set

$$W := \{T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2);\ \dim\big(\text{null}(T)\big) \geq 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$.

(5) [20 points] Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that null(T) and range(T) are invariant under S.

(6) [20 points] Suppose that (u_1, u_2, u_3, u_4) is a basis for V. Prove that

$$u_1, u_1 - u_2, u_2 - u_3, u_3 - u_4$$

is also a basis for V.