MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 8

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(1) Suppose $(V_1, \langle\cdot,\cdot\rangle_1), \dots, (V_n, \langle\cdot,\cdot\rangle_n)$ are inner product spaces. Show that
$$\langle (u_1,\dots,u_n),(v_1,\dots,v_n)\rangle := \langle u_1, v_1\rangle_1 + \dots + \langle u_n, v_n\rangle_n$$
defines an inner product on $V_1 \times \dots \times V_n$ $= \{(u_1,u_2,\dots,u_n) : u_j \in V_j \text{ where } j = 1,\dots,n\}$.

(2) Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, and define
$$f_1 = e_1 + e_2 + e_3, \quad f_2 = e_2 + e_3, \quad f_3 = e_3$$
check that (f_1, f_2, f_3) is a basis for $\mathbb{R}^3$.

(a) Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3).
(b) What do you obtain if you instead applied the Gram-Schmidt process to the basis (f_3, f_2, f_1)?

(3) Let $\mathcal{P}_2(\mathbb{R})$ be the inner product space of polynomials over $\mathbb{R}$ having degree at most two, with inner product given by
$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx, \text{ for every } f, g \in \mathcal{P}_2(\mathbb{R}).$$
Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathcal{P}_2(\mathbb{R})$ in order to produce an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$.
$f_1 \ f_2 \ f_3$

$g_1 = \frac{f_1}{\|f_1\|}$

$\|f_1\|^2 = \langle f_1, f_1\rangle = \int_0^1 f_1(x)\,dx = \int_0^1 1^2\,dx = 1$

$\boxed{g_1(x) = 1}$

$h_2(x) = f_2 - \langle f_2, g_1\rangle = x - \int_0^1 x\cdot 1\,dx$

$g_2(x) = \frac{h_2(x)}{\|h_2\|}$

(4) Prove that
$$16 \le (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$
for all positive numbers a, b, c, d. ***Hint:*** *Apply Cauchy-Schwartz Inequality.*

$n^2 \le \left(\sum_{j=1}^n a_j\right)\left(\sum_{j=1}^n \frac{1}{a_j}\right)$.

(5) Let $n \in \mathbb{Z}_+$, and let $a_1, a_2, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}$ be any collection of $2n$ real numbers. Prove that
$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \left(\sum_{k=1}^n k a_k^2\right)\left(\sum_{k=1}^n \frac{b_k^2}{k}\right).$$

$\langle u, v\rangle^2 \le \|u\|^2\|v\|^2$

$u = (a_1, \sqrt{2}a_2, \sqrt{3}a_3, \dots, \sqrt{n}a_n)$

$v = (b_1, \frac{1}{\sqrt{2}}b_2, \dots, \frac{1}{\sqrt{n}}b_n)$

(6) Let V be a finite dimensional ~~vector~~ inner product space over $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ with $P^2 = P$ and $\text{null}(P) = (\text{range}(P))^\perp$. Prove that P is an orthogonal projection.
Hint: compare and use Question 1 from Hw 7.

(7) Suppose V is finite dimensional and U is a subspace of V. Let P_U be the orthogonal projection onto U. Show that
$$P_{U^\perp} = \mathbb{1}_V - P_U$$
where $\mathbb{1} \in \mathcal{L}(V)$ is the identity map on V.

$V = U \oplus U^\perp$

$v = u_1 + u_2 \dots$

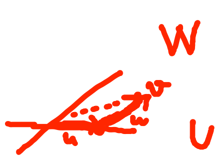
$P_U v = u_1$

$P_{U^\perp} v = u_2$

$\forall v \in V, \exists u_1, u_2 \dots$

$P_{U^\perp} v = u_2$

$(\mathbb{1} - P_U)v = v - P_U v = v - u_1 = u_2$.



$\mathbb{R}^2 = U \oplus W$

$v = u + w$ unique.

$Pv = u$.

$\boxed{Q1}$ $\langle (u_1,\ldots,u_n), (v_1,\ldots,v_n) \rangle = \sum_{j=1}^{n} \langle u_j, v_j \rangle_j$.

① Linearity in the 1^{st} slot:

$\boxed{a}$ $\langle (u_1,\ldots,u_n) + (w_1,\ldots,w_n), (v_1,\ldots,v_n) \rangle = \langle (u_1+w_1,\ldots,u_n+w_n), (v_1,\ldots,v_n) \rangle$

$$= \sum_{j=1}^{n} \langle u_j + w_j, v_j \rangle_j$$

$$= \sum_{j=1}^{n} \left(\langle u_j, v_j \rangle_j + \langle w_j, v_j \rangle_j \right)$$

$$= \sum_{j=1}^{n} \langle u_j, v_j \rangle_j + \sum_{j=1}^{n} \langle w_j, v_j \rangle_j$$

$$= \langle (u_1,\ldots,u_n), (v_1,\ldots,v_n) \rangle + \langle w's, v's \rangle.$$

$\boxed{b}$ ...

② positivity: $\langle \underbrace{-}_{u's}, \underbrace{-}_{u's} \rangle = \sum_{j=1} \langle u_j, u_j \rangle_j \geq 0$

③ pos. def. $\langle (u_1,\ldots,u_n), (u_1,\ldots,u_n) \rangle = 0 \Rightarrow \sum_{j=1}^{n} \underbrace{\langle u_j, u_j \rangle_j}_{\geq 0} = 0$

$\Rightarrow \langle u_j, u_j \rangle_j = 0 \quad \forall j = 1,\ldots,n$

$\Rightarrow u_j = 0 \quad \forall j.$

$\Rightarrow (u_1,\ldots,u_n) = 0$

④ $\langle u's, v's \rangle = \sum \langle u_j, v_j \rangle_j = \sum \overline{\langle v_j, u_j \rangle_j}$

$$= \overline{\sum \langle v_j, u_j \rangle_j} = \overline{\langle v's, u's \rangle}.$$

$\boxed{Q4}$ $16 \leq (a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \cdots + \frac{1}{d}\right)$

$u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$, $\|u\|^2 = (a+b+c+d)$.

$v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$, $\|v\|^2 = \frac{1}{a} + \cdots + \frac{1}{d}$

$|\langle u,v \rangle|^2 = 16.$ ...

$|\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2$

$u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$

$\|u\|^2 = \cancel{\sqrt{a^2+b^2+c^2+d^2}}$

$\boxed{Q6}$ $P^2 = P$, $\quad \text{null}(P) = (\text{range}(P))^\perp$

$$V = \text{null}(P) \oplus (\text{null}(P))^\perp$$

$$V = \text{null}(P) \oplus \text{range}(P).$$

<u>claim</u>; P is an $\perp$ proj onto $\text{range}(P)$.

let $v \in V$, $\exists!\, v_1 \in \text{null}(P)$ & $v_2 \in \text{range}(P)$;

$$v = v_1 + v_2$$

(Apply P) $\quad Pv = \underbrace{Pv_1}_{=0} + Pv_2 = Pv_2$

$v_2 \in \text{range}(P)$ $\quad \exists\, w \in V;\ Pw = v_2$

$P^2 = P$ $\quad \Rightarrow P^2 w = Pv_2$

$\quad \Rightarrow Pw = Pv_2 = v_2$

$Pv = Pv_2 = v_2$

∎

$V = U \oplus U^\perp$

$v = u_1 + u_2$

$Pv = u_1$.