MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 7

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(1) Let V be a finite dimensional vector space over the field $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ has the properties that $P^2 = P$. Prove that $V = \text{null}(P) \oplus \text{range}(P)$.

(2) Solve for x the following equation

$$\det\left(\begin{bmatrix} x & -1 \\ 3 & 1-x \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{bmatrix}\right)$$

(3) Use the definition of the determinant to derive the formula for the determinant of a 4×4 matrix.

(4) Find the determinant of the following matrix

$$\begin{bmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) - \cos(\theta) & \sin(\theta) + \cos(\theta) & 1 \end{bmatrix}$$

(5) Suppose $(V, \langle \cdot, \cdot \rangle)$ is an inner product space. Let $T \in \mathcal{L}(V)$ is injective. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 := \langle Tu, Tv \rangle, \text{ for all } u, v \in V.$$

Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V.

(6) Suppose that V is a real inner product space
   (a) Show that if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.
   (b) Use part (a) to show that the diagonals of a rhombus are perpendicular to each other.

(7) Let V be a finite dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that

$$\langle u, v \rangle = \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right).$$

(8) Suppose the $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for all $v \in V$. Prove that $(T - 2\mathbb{1})$ is invertible.