MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 6

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SUMMER 2020

- **Submit questions 2 for grading.**
- **<u>Due on:</u> Tuesday June 23, 2020.**
- **You will have a quiz from the following questions on Tuesday June 23, 2020.**

In the following, U, V, and W are finite dimensional vector spaces over a field $\mathbb{F}$.

(1) Let $T \in \mathcal{L}(V)$ and let U_1 and U_2 be two T-invariant subspaces of V.
  (a) Prove that $U_1 + U_2$ is T-invariant.
  (b) Prove that $U_1 \cap U_2$ is T-invariant.

(2) Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null} S$ and $\operatorname{range} S$ are T-invariant.

(3) Suppose that V is a finite dimensional complex vector space nd $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension k for each $k = 1, \ldots, \dim V$.

(4) Let $n \in \mathbb{Z}_+$ be a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ be defined by

$$T(x_1, \ldots, x_n) = (x_1 + \ldots + x_n, \ldots, x_1 + \ldots + x_n)$$

for every $x_1, x_2, \ldots, x_n \in \mathbb{F}$. Compute the eigenvalues and associated eigenvectors for T.

(5) Suppose that $V = U \oplus W$, where U and W are subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for every $u \in U$ and $w \in W$. Find all eigenvalues and eigenvectors of P.

(6) Suppose $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(V)$ invertible. Show that T and $S^{-1}TS$ have the same eigenvalues.

(7) Suppose that $T \in \mathcal{L}(V)$ is invertible. Prove that

$$\operatorname{null}(T - \lambda \mathbb{1}) = \operatorname{null}(T^{-1} - \frac{1}{\lambda}\mathbb{1})$$

for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

(8) Suppose that $T \in \mathcal{L}(V)$ has the property that every $v \in V$ is an eigenvector for T. Prove that T must then be a scaler multiple of the identity function on V.

(9) Let V be a finite dimensional over $\mathbb{C}$, $T \in \mathcal{L}(V)$, and $p(z) \in \mathbb{C}[z]$ be a polynomial. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of the linear operator $p(T) \in \mathcal{L}(V)$ if and only if T has an eigenvalue $\mu \in \mathbb{C}$ such that $p(\mu) = \lambda$.