

MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 4

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SUMMER 2020

- **Submit question 2 for grading.**
- **Due on: Tuesday June 16, 2019.**
- **You will have a quiz from the following questions on Tuesday June 16, 2019.**

(1) Suppose $b, c \in \mathbb{R}$. Define $T : \mathbb{R}[x] \rightarrow \mathbb{R}^2$ by

$$Tp = \left(3p(4) + 5p'(6) + bp(1)p(2), \int_{-1}^2 x^3 p(x) dx + c \sin(p(0)) \right).$$

Show that T is linear if and only if $b = c = 0$.

- (2) Suppose $T \in \mathcal{L}(V, W)$ and v_1, \dots, v_m is a list of vectors in V such that (Tv_1, \dots, Tv_m) is linearly independent list of vectors in W . Prove that v_1, v_2, \dots, v_m are linearly independent.
- (3) Give an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$f(\alpha v) = \alpha f(v)$$

for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^2$, but f is not linear.

(4) Check that the function $f : \mathbb{C} \rightarrow \mathbb{C}$ defined as

$$f(z) = \operatorname{Re} z$$

satisfies

$$f(u + v) = f(u) + f(v)$$

for all $u, v \in \mathbb{C}$, but f is not linear.

(5) Suppose V is a vector space and $S, T \in \mathcal{L}(V)$ are such that

$$\operatorname{range} S \subseteq \operatorname{null} T.$$

Prove that $(ST)^2 = 0$.

(6) Show that

$$U := \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4); \dim(\operatorname{null} T) > 2\}$$

is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$. *Hint: Give an example of two linear maps S and T in U such that $S + T$ is not in U .*

(7) Give an example of a linear map $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

(8) Prove that there does not exist a linear map $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that

$$\operatorname{range} T = \operatorname{null} T.$$

(9) Suppose that V is finite dimensional vector space and let $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu; u \in U\}$.