MATH 413/513 (LINEAR ALGEBRA)
HOMEWORK 3

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SUMMER 2020

- **Submit question 7 for grading.**
- **Due on: Monday June 8, 2020.**
- **You will have a quiz from the following questions on Monday June 8, 2020.**

(1) Show that the vectors $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, and $v_3 = (2, -1, 1)$ are linearly independent in $\mathbb{R}^3$. Write $v = (1, -2, 5)$ as a linear combination of v_1, v_2, and v_3.

(2) Define $U = \text{span}(u_1, u_2, \ldots, u_n)$. Suppose $v \in U$, prove
$$U = \text{span}(v, u_1, u_2, \ldots, u_n).$$

(3) Suppose that $v_1, v_2, \ldots, v_n$ is a linearly independent set of vectors in V. Given any $w \in V$ such that the set of vectors
$$v_1 + w, v_2 + w, \ldots, v_n + w$$
is linearly dependent, prove that $w \in \text{span}(v_1, v_2, \ldots, v_n)$.

(4) Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that
$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$
is also a basis of V.

(5) Find the dimension of the following subspace of $\mathbb{R}^4$
$$U = \{(x_1, x_2, x_3, x_4) |\ x_4 = x_1 + x_2\}.$$

(6) Let $\dim(V) = n$ for some $n \in \mathbb{Z}_+$. Prove that there are n one-dimensional subspaces $U_1, U_2, \ldots, U_n$ of V such that
$$V = U_1 \oplus U_2 \oplus \ldots \oplus U_n.$$

(7) Let $U = \{p \in \mathcal{P}_4[\mathbb{F}] :\ p(6) = 0\}$.
   (a) Find a basis of U.
   (b) Extend the basis in part (a) to a basis of $\mathcal{P}_4[\mathbb{F}]$.
   (c) Find a subspace W of $\mathcal{P}_4[\mathbb{F}]$ such that $\mathcal{P}_4[\mathbb{F}] = U \oplus W$.

(8) Let U and V be four-dimensional subspace of $\mathbb{R}^7$. Prove that $U \cap V \neq \{0\}$.

(9) Suppose $p_0, p_1, \ldots, p_m \in \mathcal{P}[\mathbb{F}]$ are such that each p_j has degree j. Prove that $p_0, p_1, \ldots, p_m$ is a basis of $\mathcal{P}_m[\mathbb{F}]$.

(10) Suppose that $p_0, p_1, \ldots, p_m \in \mathcal{P}_m[\mathbb{F}]$ satisfy $p_j(1) = 0$. Prove that $(p_0, p_1, \ldots, p_m)$ is a linearly dependent list of vectors in $\mathcal{P}_m[\mathbb{F}]$.