MATH 413/513 (LINEAR ALGEBRA)

HOMEWORK 8 - SUMMER 2018

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<u>Due on:</u> Tuesday 06-26-2018.

(1) Let V be a finite dimensional vector space over $\mathbb{C}$ with $T \in \mathcal{L}(V)$. Prove that, for each $k = 1, 2, \ldots, \dim(V)$, there is a T-invariant subspace U_k of V such that $\dim(U_k) = k$.

(2) Let A be an invertible matrix in $\mathbb{R}^{n\times n}$, suppose that $A^2 - 2A = 0$, find $\det(2A^T A^2)$.

(3) Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$, and define

$$f_1 = e_1 + e_2 + e_3, \quad f_2 = e_2 + e_3, \quad f_3 = e_3$$

(a) Apply the Gram-Schmidt process to the basis (f_1, f_2, f_3).
(b) What do you obtain if you instead applied the Gram-Schmidt process to the basis (f_3, f_2, f_1)?

(4) Let $\mathbb{R}_2[x]$ be the inner product space of polynomials over $\mathbb{R}$ having degree at most two, with inner product given by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx, \text{ for every } f, g \in \mathbb{R}_2[x].$$

Apply the Gram-Schmidt procedure to the standard basis $\{1, x, x^2\}$ for $\mathbb{R}_2[x]$ in order to produce an orthonormal basis for $\mathbb{R}_2[x]$.

(5) Let $n \in \mathbb{Z}_+$, and let $a_1, a_2, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ be any collection of $2n$ real numbers. Prove that

$$\left(\sum_{k=1}^n a_k b_k\right)^2 \le \left(\sum_{k=1}^n k a_k^2\right)\left(\sum_{k=1}^n \frac{b_k^2}{k}\right).$$

Hint: Use Cauchy-Schwarz inequality with the "correct" choice of vectors.

(6) Let V be a finite dimensional inner product space over $\mathbb{R}$. Given $u, v \in V$, prove that

$$\langle u, v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right).$$

(7) Let V be a finite dimensional vector inner product space over $\mathbb{F}$, and suppose that $P \in \mathcal{L}(V)$ with $P^2 = P$ and $\text{null}(P) = (\text{range}(P))^\perp$. Prove that P is an orthogonal projection.