Test 2 **Linear Algebra MA 413/513** **June 15, 2018**

Name: _____ **Signature:** _____

SHOW ALL YOUR WORK!

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1. [15 points] Suppose that U is a 3-dimensional subspace of $\mathbb{R}^8$ and that $T \in \mathcal{L}(\mathbb{R}^8, \mathbb{R}^5)$ such that $\text{null}(T) = U$. Prove that T is surjective.

2. [15 points] V is a finite dimensional vector space over a field $\mathbb{F}$. Let $T \in \mathcal{L}(V)$, prove that if U_1 and U_2 are two invariant subspaces under T then

   (a) $U_1 + U_2$ is invariant under T.

   (b) $U_1 \cap U_2$ is invariant under T.

3. [15 points] Let V and W be vector spaces over $\mathbb{F}$, and suppose $T \in \mathcal{L}(V, W)$ is surjective. Given that $V = \text{span}(v_1, \ldots, v_n)$, prove that $W = \text{span}(Tv_1, \ldots, Tv_n)$.

4. [15 points] Let V be a finite dimensional vector space over a field $\mathbb{F}$, and let $S, T \in \mathcal{L}(V)$. Prove that $T \circ S$ is invertible if and only if both S and T are invertible.

5. [15 points] Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $T_j \in \mathcal{L}(V)$ for $j = 1, 2, \ldots, k$. Suppose that there is a non-zero vector $v \in V$ and a constant $\lambda \in \mathbb{F}$ such that

$$(T_1 - \lambda \mathbb{1})(T_2 - \lambda \mathbb{1}) \ldots (T_k - \lambda \mathbb{1})v = 0$$

Prove that there exists a $j \in \{1, 2, \ldots, k\}$ such that λ is an eigenvalue of T_j.

6. [15 points] Let V be a finite dimensional vector space over the field $\mathbb{F}$, and let $\phi \in \mathcal{L}(V, \mathbb{F})$. Suppose $u \in V$ is not in $\text{null}(\phi)$. Prove that

$$V = \text{null}(\phi) \oplus \text{span}\{u\}.$$

7. [15 points] Let V be a finite dimensional vector space over the field $\mathbb{F}$, and let $T \in \mathcal{L}(V)$ such that every $v \in V$ is an eigenvector for T. Prove that there exists an $\alpha \in \mathbb{F}$ such that $T = \alpha \mathbb{1}$.