Test 1 **Linear Algebra MA 413/513** **May 29, 2018**

Name: ______________________ **Signature:** ______________________

SHOW ALL YOUR WORK!

In all of the following questions, V is a finite dimensional vector space over a field $\mathbb{F}$.

1. [10 points] Solve the equation $z^3 + 8i = 0$, where $z \in \mathbb{C}$.

2. [10 points] Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 \oplus W,$$

then $U_1 = U_2$.

3. [20 points] Solve one of the following two questions as directed

- **For 413 students**
  Let U_1 and U_2 be two subspaces of V.
  (a) Show that $U_1 \cap U_2$ is a subspace of V.
  (b) Show that $U_1 + U_2$ is a subspace.

- **For 513 students**
  Let $\mathbb{F}[z]$ be the vector space of all polynomials with coefficients from the field $\mathbb{F}$. Let

  $$W = \mathbb{F}[z] \times \mathbb{F}^2 = \{(q, u);\ q \in \mathbb{F}[z], u \in \mathbb{F}^2\}.$$

  Equip W with a vector addition $(+)$ and a scalar multiplication $(\cdot)$ in such a way that W becomes a vector space, and prove that $W(+, \cdot)$ is a vector space over $\mathbb{F}$.

4. [20 points] Let W_1 and W_2 be two subspaces of V,
prove that $V = W_1 \oplus W_2$ if and only if the following conditions hold

(1) $V = W_1 + W_2$ and

(2) If $0 = w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$, then $w_1 = w_2 = 0$.

5. [20 points] Suppose that v_1, v_2, v_3, v_4, v_5 is a linearly independent list in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_5$$

is linearly independent.

6. [20 points] In each of the following subsets of $\mathcal{C}(\mathbb{R})$, prove or disprove that it is a subspace.

   (a) The set of all $f \in \mathcal{C}(\mathbb{R})$ such that $f(x) \leq 0$ for all $x \in \mathbb{R}$.

   (b) The set of all $f \in \mathcal{C}(\mathbb{R})$ such that $\int_0^1 f(x)\, dx = 1$.

   (d) The set of all $f \in \mathcal{C}(\mathbb{R})$ such that $f(2)$ is an integer.

   (c) The set of all $f \in \mathcal{C}(\mathbb{R})$ such that for some $\alpha, \beta, \gamma \in \mathbb{R}$

$$\alpha \frac{d^2 f}{dt^2} + \beta \frac{df}{dt} + \gamma f = 0.$$

7. [10 points] Solve one of the following two questions as directed

- **For 413 students**
  Prove that $U = \{(x_1, x_2, x_3) \in \mathbb{F}^3 |\ x_3 = x_1 + x_2\}$ is a subspace from $\mathbb{F}^3$.

- **For 513 students**
  Consider the vector space $V = \mathcal{C}(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$.
  Let $U_o \subset V$ and $U_e \subset V$ be the sets of odd and even continuous functions respectively.
  Recall: $f : \mathbb{R} \to \mathbb{R}$ is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, and is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.
  (a) Show that for any $f \in \mathcal{C}(\mathbb{R})$,
  $$g(x) = \frac{f(x) + f(-x)}{2} \text{ is even, and } h(x) = \frac{f(x) - f(-x)}{2} \text{ is odd.}$$
  (b) Prove that U_o and U_e are subspaces of V.
  (c) Show that $V = U_o \oplus U_e$.