

ENTANGLEMENT IN DISORDERED QUANTUM XY CHAINS

by

HOUSSAM M. A. ABDUL RAHMAN

GÜNTER STOLZ, COMMITTEE CHAIR
TIMOTHY FERGUSON
RYOICHI KAWAI
BORIS KUNIN
SHANNON STARR

A DISSERTATION

Submitted to the faculty of the University of Alabama at Birmingham,
in partial fulfillment of the requirements of the degree of
Doctor of Philosophy

BIRMINGHAM, ALABAMA

2016

Copyright by
HOUSSAM M. A. ABDUL RAHMAN
2016

ENTANGLEMENT IN DISORDERED QUANTUM XY CHAINS

HOUSSAM M. A. ABDUL RAHMAN

APPLIED MATHEMATICS

ABSTRACT

We consider a class of quantum disordered XY chains where the eigenfunction correlator localization of the corresponding effective single particle Hamiltonian is satisfied. We consider the dynamics of the XY chain and we show that, starting from a broad class of product initial states, bipartite entanglement remains bounded for all times. Corollaries include area laws for the entanglement entropy of eigenstates and for the dynamics of any up-down configuration initial state. For the disordered isotropic XY chain we derive bounds for the particle number transport and prove that the expected number of particles that can penetrate out of a block of spins decays in distance. These results demonstrate the fact that the disordered XY chains are fully many-body localized.

DEDICATION

· *To my parents who constantly loved, supported, and encouraged me.*

· *To my beloved wife, Lama, who has stood in my side and supported me with continuous love and patience.*

· *To my sweet little daughters.*

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor Professor Günter Stolz, who gave me a lot of his time and efforts. I am extremely fortunate to have the opportunity to work closely to him. Above all what I have learned from him, he is a constant provider of support and care.

I am thankful to the Professors in the Department of Mathematics at UAB who prepared me for my next phase of my career, I start with Dr. Yulia Kerpeshina and Dr. Rudi Weikard who also supported me to get a scholarship during my PhD study. And I acknowledge the members of the Probability and Math-Physics seminar: Dr. Paul Jung, Dr. Yulia Karpeshina, Dr. Nandor Simanyi, Dr. Shannon Starr, Dr. Günter Stolz, and Dr. Rudi Weikard.

I would also like to thank Dr. Ryoichi Kawai from the Physics Department at UAB for his time in the interesting reading courses he offered for me.

I am thankful to my thesis committee: Dr. Timothy Ferguson, Dr. Ryoichi Kawai, Dr. Boris Kunin, and Dr. Shannon Starr.

I would like to thank Dr. Bruno Nachtergaele from the University of California at Davis and Dr. Robert Sims from the University of Arizona for the helpful discussions and for their support.

I thank the Department of Mathematics at the Jordan University of Science and Technology. I would like particularly to thank Dr. Mahmoud Alrefaei for introducing me to mathematical research as undergraduate.

Finally, I thank my parents and my wife for their unwavering support throughout my graduate study.

TABLE OF CONTENTS

ABSTRACT	iii
DEDICATION	iv
ACKNOWLEDGEMENTS	v
LIST OF FIGURES	viii
INTRODUCTION	1
CHAPTER 1. TOWARDS MANY-BODY LOCALIZATION	2
1.1. Motivation	2
1.2. A short survey of known results	4
1.3. The main results	6
CHAPTER 2. BASIC CONCEPTS FROM QUANTUM MECHANICS	7
2.1. Pure and mixed states of a quantum system	7
2.2. Dynamics	10
2.3. Coupling quantum systems and entanglement	11
2.4. Entropy	14
2.5. Partial traces	17
2.6. Entanglement and von Neumann entanglement entropy	21
CHAPTER 3. THE THEORY OF FINITE FERMIONIC SYSTEMS	23
3.1. Fermionic operators	23
3.1.1. Jordan-Wigner Fermionic system	28
3.1.2. Bogoliubov transformation	30
3.2. Quasi-free states	34
3.2.1. Diagonal states	37
3.2.2. Reduced states of quasi-free states	43
3.2.3. Tensor product of quasi-free states	47
3.3. Correlation matrices	53
3.3.1. Correlation matrix of a reduced state	58
3.3.2. Correlation matrix of tensor product of states	58
3.4. Entropy of quasi-free states	59
3.4.1. Diagonalizing correlation matrices	59
3.4.2. Correlation matrices of quasi-free states	64
CHAPTER 4. PARTICLE NUMBER TRANSPORT IN THE ISOTROPIC XY CHAIN	71
4.1. The model and main assumptions	71
4.2. Reduction to the single particle Hamiltonian	72

4.3. Particle number operator	74
4.4. Weak particle number transport	76
4.5. Proof of main results	79
CHAPTER 5. ENTANGLEMENT DYNAMICS OF DISORDERED XY CHAINS	84
5.1. The model and main assumptions	84
5.2. Reduction to the single particle Hamiltonian	85
5.3. Problem setup and results	89
5.4. Higher moments	92
5.5. Applications	94
5.6. Proofs of results	95
CHAPTER 6. A BRIEF DISCUSSION OF CURRENT AND FUTURE WORK	111
LIST OF REFERENCES	113
APPENDIX A. BASICS OF PFAFFIANS	118
APPENDIX B. SOME INEQUALITIES	120
B.1. Peierls-Bogoliubov inequality	120
B.2. Other inequalities	121

LIST OF FIGURES

1.1 Vertical lines stand for the one-particle configuration space, quantum interaction is represented by the horizontal arrows.	2
1.2 Gray connections are void and are turned on for $t > 0$	6
3.1 Local Jordan-Wigner operators when $n = 7$ and $\ell = 3$	30
4.1 Subregions Λ_0 and S over the initial state φ , here Λ_0 occupies the left end of the chain.	78
5.1 An example with $m = 4$, the subinterval Λ_0 is chosen to be $\Lambda_2 \cup \Lambda_3$	91

INTRODUCTION

The purpose of this dissertation is to present new results about some manifestations of Many-Body Localization (MBL) in a class of quantum XY chains. In particular, we present results about the entanglement and particle number transport. Most of the thesis is based on materials from [[ARS15](#), [ARNSS16](#)]. The main results are presented along with their proofs in Chapters [4](#) and [5](#).

Throughout this thesis, we assume that the reader has a certain level of knowledge of operator theory. In particular, topics such as spectral theory for compact operators and bounded operators acting on the tensor product of separable Hilbert spaces.

Chapter [1](#) draws the big picture and present the motivation for our contribution. Chapter [2](#) covers the basic background about the mathematical formulation of quantum mechanics needed for our work. In Chapter [3](#), we present our approach to the theory of finite Fermionic systems, where the study of the XY chain reduces to the study of Fermionic systems. In Chapter [4](#), results about particle number transport for the isotropic XY model are presented. Chapter [5](#) includes the dynamical entanglement results of the disordered XY model. Finally, we discuss briefly some future plans in Chapter [6](#).

CHAPTER 1

Towards Many-Body Localization

1.1. Motivation

In this thesis, we are interested in a better understanding of mathematical characterizations of *many-body localization* (MBL) in interacting quantum systems. This phenomenon has recently received strong attention in theoretical physics and quantum information theory, see e.g. [BPM12, BN13, BO07, CRF⁺11, FWB⁺15, GMP05, HNO14, SKG⁺14, SPA13, MSA13, VA13, ZPP08].

MBL lets a system act as an insulator due to the presence of disorder. Systems that are many-body localized can locally remember forever information about their initial conditions, and are thus of interest for possibilities of storing quantum information [NH15]. MBL is generally described in the physics literature as the absence of *thermalization* or *self-equilibration* in a quantum many-body system. But despite the great efforts to address this phenomena, it is not fully understood and a precise definition is still lacking. MBL is currently a subject of debate [BAA06, PH10, OH07, RH15, RHMS16, ARH16, GE16].

In what follows, we provide a heuristic simple argument explaining MBL. For the sake of simplicity, we will consider a one dimensional symbolic picture. Consider a chain Λ of n particles, where n is an arbitrarily large positive integer.

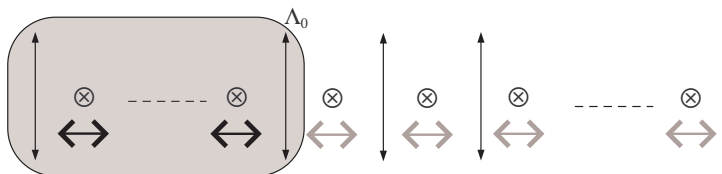


FIGURE 1.1. Vertical lines stand for the one-particle configuration space, quantum interaction is represented by the horizontal arrows.

Vertical lines in Figure 1.1 represent the configuration space of a single particle, e.g. an electron. Bold horizontal arrows demonstrate the interaction between particles. By distinguishing a subregion Λ_0 (shaded), pairwise interactions can also be regarded as an interaction between Λ_0 and the rest of the chain. While the well understood one-body localization, see for example [Sto11], is for the (localized) vertical motion of the particle in time, MBL is all about the horizontal direction, it is the so-called absence of (or weak) information transport, precisely between Λ_0 and $\Lambda \setminus \Lambda_0$. This can be restated as “localization of information” in Λ_0 .

We conclude that a localized many-body system, in suitable regimes such as weak interaction or large disorder, should have properties in a way similar to a system consisting of non-interacting local Hamiltonians, supported, in our case, on Λ_0 and its complement, respectively. We will focus here on such criteria which by now are well accepted to be necessary characteristics of MBL and on studying these criteria for relatively simple models where they can be proven rigorously. The eigenstates of non-interacting systems are product states and thus have vanishing entanglement and spatial correlations. Also, its dynamics are trivial with no information propagating between particles. For an interacting system in the MBL phase one thus expects rapid decay of correlations and small entanglement of eigenstates, as well as absence of information transport (i.e. no or slow propagation of particle group waves).

Let us stress here that the term MBL should generally be reserved for properties which hold uniformly in the number of particles in the system (e.g. in the sense that relevant constants are bounded in the particle number). In this sense, many-body localization is to be distinguished from few-body localization, such as the known rigorous results for the N -particle Anderson model [CS09a, CS09b, AW09, KN13], which do not yet allow uniform control in the number of electrons. In particular, results expected by physicists for the many-body Anderson model, such as MBL at low electron density or weak interaction strength, e.g. [BAA06, GMP05], can not yet be shown rigorously.

1.2. A short survey of known results

To gain further insight in the nature of MBL, it is useful to study simple model systems. A number of interesting discoveries have been made in recent years through numerical studies of the quantum Ising chain, the XY chain, and the XXZ chain [ZPP08, BPM12].

Rigorous mathematical results on localization properties of disordered many-body systems are so far essentially restricted to two models: Disordered harmonic oscillator systems [NSS12, NSS13] and the XY spin chain in random field [KP90, HSS12, SW16b, PS14, GL16]. A recent work with rigorous mathematical results is about the Tonks-Girardeau gas, which is a continuum analogue of the XY spin chains [SW16a]. These models are equivalent to free Boson systems and free Fermion systems, respectively, and thus can be studied in terms of an effective one-particle Hamiltonian. As a consequence, it is possible to deduce results on MBL from known localization properties of one-particle Hamiltonians such as the Anderson model. Of course, a long term goal must be to develop methods which allow to go beyond these simple models. In particular, an important challenge is to develop mathematical methods to study the disordered XXZ or Heisenberg chain, which reduce to the physically more interesting situation of *interacting* Fermion systems, see e.g. [BPM12, BN13, OH07, PH10, SPA13] for numerical results. Some progress towards understanding MBL for larger classes of spin systems in the presence or strong disorder has been made in [Imb16]. Mathematically, describing the phenomenon of many-body localization and fully proving it for important classes of physical examples is a wide open field.

For oscillator systems the reduction to an effective one-particle Hamiltonian is rather straightforward. In particular, the reduction works in any dimension and does not affect locality properties of the system. This has allowed to verify a quite complete list of MBL properties for disordered oscillator systems, including a zero-velocity Lieb-Robinson bound on information transport as well as exponential decay

of correlations and area-law-type entanglement bounds for ground and thermal states [NSS12, NSS13] (in each case requiring disorder averaging).

The reduction of the XY model via the Jordan-Wigner transform to a free Fermion system is only possible for a one-dimensional chain of spins and, as an additional difficulty, introduces non-locality. Therefore rigorously known MBL properties for the XY model are more limited.

A first contribution was made in [KP90] which established exponential decay of certain ground state correlations for the isotropic XY chain in random exterior field. Using localization of eigenfunction correlators of the underlying one-particle Hamiltonian, Sims and Warzel [SW16b] found exponential decay of stationary as well as time-dependent correlations for larger classes of states, including general eigenstates as well as thermal states, for systems such as the XY chain which can be mapped to free Fermions. Also, it was proved that the bipartite entanglement entropy of its ground state satisfies an area law [PS14], and it admits a well defined asymptotic form with probability one [EPS16]. Localization of eigenfunction correlators is also the key property of the effective Hamiltonian used in [HSS12] to show a zero-velocity Lieb-Robinson bound for the XY chain in random field, again after disorder averaging. An argument in [HSS12], valid for a very general class of quantum spin systems shows that this implies exponential decay of ground state correlations up to a logarithmic correction in the size of the ground state gap (the more specific arguments used in [SW16b] show that no such correction is required for the random XY chain). This in turn, by another general observation, leads to an area law for the ground state entanglement [BH13, BH15]. Note, however, that [BH13, BH15] consider deterministic systems and that some further analysis would be required to show that the relation “exponential decay of correlations implies area law” carries over to the disorder averaged quantities.

1.3. The main results

In this thesis, we study the entanglement dynamics in a class of disordered quantum XY chains in the dynamical localization regime. We prove that for a large class of product initial conditions the bipartite entanglement satisfies a constant bound, independent of time and system size, i.e., the same type of “Area Law” bound Hastings proved for gapped ground states in one dimension [Has07]. In particular, we consider a quantum XY chain that has few missing connections between the adjacent spins at time $t = 0$. The eigenstates of this chain are product states. For $t > 0$ we establish the interactions and we consider the Schrödinger dynamics under the connected chain, of the initial eigenstates, see Figure 1.2.



FIGURE 1.2. Gray connections are void and are turned on for $t > 0$.

We prove that the bipartite entanglement of this dynamics is bounded uniformly in time and the system size. As a corollary, we prove an area law for the disorder averaged entanglement of *all* eigenstates of the XY chain in random field. This is in agreement with the numerical predictions of [BPM12] and it confirms the doubts expressed in [CMCF06] that the observed logarithmic growth of entanglement for short times holds for all times. Disordered XY chains show many of the features usually associated with MBL [BAA06, PH10]. By exhibiting complete localization, the entanglement dynamics of the XY chain, however, appears to deviate from the logarithmic growth in time generically expected and observed numerically in other model systems [CMCF06, ZPP08, BPM12].

Moreover, in the isotropic XY chain, sometimes referred to as the XX chain, with a random field in the Z direction, the particle number is conserved. In this case we prove that the number of particles moving out a subregion (reservoir) is decaying in distance. This is in stark contrast with [ARRS08, ARRS99, Oga02].

CHAPTER 2

Basic Concepts From Quantum Mechanics

In this chapter, we present the necessary background from the mathematical formulation of quantum mechanics which will be needed for our results. We made no efforts to present a comprehensive presentation for all basics of quantum mechanics and our choice for the topics is very biased toward the needed concepts in the rest of the thesis. For example, any quantum mechanical textbook starts with stressing the axioms. However, in this chapter we give priority to the ingredients such as states and observables and mention the axiom as an extra piece of information. As a result, an attentive reader will notice that the “measurement” axiom of quantum mechanics is not mentioned because we don’t need it in this thesis. For complete presentations, we refer the reader to references such as [CTDL91, SN11, Pet08].

We assume that the reader is familiar with basic Hilbert space theory, where we will consider only separable Hilbert spaces, see for example [Wei80, RS75]. We note here that we define the inner product $\langle f, g \rangle$ to be linear in the second parameter and conjugate linear in the first parameter.

2.1. Pure and mixed states of a quantum system

To each quantum mechanical system a complex Hilbert space \mathcal{H} is associated. This is the first axiom of quantum mechanics. A unit vector $\phi \in \mathcal{H}$ is called a *pure state* of the quantum system. Unit vectors ϕ and ψ are representing the same pure state if they are equal up to a phase, this means that there exists a constant $z \in \mathbb{C}$ with $|z| = 1$ and $\phi = z\psi$.

The *density matrix* of a pure state $\phi \in \mathcal{H}$ is defined by the rank one projection of ϕ ,

$$P_\phi := |\phi\rangle\langle\phi|,$$

where we used here the convenient physical notation equivalent to say that for any $f \in \mathcal{H}$, $P_\phi f := \langle\phi, f\rangle\phi$.

Let ϕ and ψ are two unit vectors, then they are representing the same state if and only if $P_\phi = P_\psi$. This can be seen using the following argument: if $\phi = z\psi$ with $|z| = 1$ then

$$P_\phi = |\phi\rangle\langle\phi| = |z|^2|\psi\rangle\langle\psi| = P_\psi.$$

and if $P_\phi = P_\psi$ then for any $f \in \mathcal{H}$ and $\langle\phi, f\rangle \neq 0$ then

$$|\phi\rangle\langle\phi|f = |\psi\rangle\langle\psi|f \Rightarrow \phi = \frac{\langle\psi, f\rangle}{\langle\phi, f\rangle}\psi$$

and since $\|\phi\| = \|\psi\| = 1$, then ϕ and ψ are equal up to a phase, hence they are representing the same state of the system.

Because of that, the notion “pure state” will be used to describe a unit vector or its corresponding density matrix, the explicit meaning will be understood from the context.

In the following we will define “mixed states”, where we need to use the theory for trace class operators. We list below some basic definitions and facts without proofs, we refer the reader to operator theory textbooks, for example Chapter 7 in [Wei80].

The *Singular Value Decomposition* (SVD) for compact operators refers to the following: let $T \in \mathcal{B}_\infty(\mathcal{H})$, then the positive self-adjoint operator $|T| = (T^*T)^{\frac{1}{2}}$ has nonzero eigenvalues $s_1 \geq s_2 \geq \dots > 0$. It is known that these eigenvalues are either finite or converge to 0. Additionally, there exists orthonormal sequences $\{f_j\}$ and $\{g_j\}$ such that

$$Tf = \sum_j s_j \langle f_j, f \rangle g_j, \quad \text{or in physics notation} \quad T = \sum_j s_j |f_j\rangle\langle g_j|.$$

T is a *trace class* operator, denoted $T \in \mathcal{B}_1(\mathcal{H})$, if

$$\|T\|_1 := \sum_j s_j < \infty.$$

And T is a *Hilbert-Schmidt* operator, denoted by $T \in \mathcal{B}_2(\mathcal{H})$, if

$$\|T\|_2 := \sum_j s_j^2 < \infty.$$

It is clear that $\mathcal{B}_1(\mathcal{H}) \subset \mathcal{B}_2(\mathcal{H})$, and in addition to that it is known that $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}_\infty(\mathcal{H})$. For $T \in \mathcal{B}_1(\mathcal{H})$, the trace of T is defined as

$$\text{Tr}[T] := \sum_j \langle \phi_j, T\phi_j \rangle$$

where $\{\phi_j\}$ is an orthonormal basis of \mathcal{H} . Furthermore, it is known that $\text{Tr}[T]$ is independent of the choice of orthonormal basis, and the series in the definition is absolutely convergent.

Note that if $T \in \mathcal{B}_1(\mathcal{H})$ and $T \geq 0$ (in particular, self-adjoint), then from the spectral theorem for compact self-adjoint operators we have that there exists an orthonormal basis (ONB) $\{\phi_i\}$ of eigenvectors of T which correspond to the eigenvalues λ_j such that

$$T = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|.$$

Thus, since $T \geq 0$, we have that $T = |T|$. Thus, for all j we have $0 \leq \lambda_j = s_j$. Thus,

$$\text{Tr}[T] = \sum_j \langle \phi_j, T\phi_j \rangle = \sum_j \lambda_j = \sum_j s_j = \|T\|_1.$$

Now we can define mixed states

DEFINITION 1. *A mixed state is a non-negative trace class operator $\rho \in \mathcal{B}_1(\mathcal{H})$ with*

$$\text{Tr}[\rho] = \|\rho\|_1 = \sum_j \lambda_j = 1$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ are non-negative eigenvalues of ρ (with multiplicity).

Note that the mixed states can be decomposed as

$$\rho = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j|. \quad (2.1)$$

We see that pure states are a special case of mixed states. We can take the pure state $P_\phi = |\phi\rangle\langle\phi|$ and represent it as a mixed state by setting $\phi_1 = \phi$ and $\{\phi_j\}_{j=2}^\infty$ as an arbitrary orthonormal basis of $\{\phi\}^\perp$. Then set $\lambda_1 = 1$ and $\lambda_j = 0$ for $j \geq 2$. And since P_ϕ is an orthogonal projection i.e. $P_\phi^2 = P_\phi$, we can say that if $\rho \in \mathcal{B}_1(\mathcal{H})$ is non-negative with trace one and $\rho^2 = \rho$ then it is a pure state. In the following, the notion “state” will be used to denote a generic mixed/pure state, and the exact meaning will be understood from the context.

2.2. Dynamics

The “observables” axiom of quantum mechanics relates any physical quantity attached to the quantum system to a self-adjoint operator acting on \mathcal{H} . Such self-adjoint operators are called *observables* of the system.

One observable of the system \mathcal{H} represents the total energy of the system. This observable is called the *Hamiltonian* of the system, let us denote it by H , it controls the way the system is evolving with time, assuming that the system is closed. The “dynamics” axiom of quantum mechanics implies that if the system is in an initial state ϕ (at time $t = 0$), then the state of the system at time t is then given by

$$\phi_t := e^{-itH}\phi \quad (2.2)$$

and thus the evolution of $\rho = |\phi\rangle\langle\phi|$ is given by

$$\rho_t = e^{-itH}\rho e^{itH}. \quad (2.3)$$

This is the so-called *Schrödinger* evolution of ρ .

In quantum mechanics, the expected value of an observable A at a general state ρ is given by

$$\langle A \rangle_\rho := \text{Tr}[A\rho]. \quad (2.4)$$

Note that in the case where ρ is a pure state, i.e. $\rho = |\phi\rangle\langle\phi|$ for some unit vector $\phi \in \mathcal{H}$, then (2.4) reads

$$\langle A \rangle_\rho = \langle \phi, A\phi \rangle$$

Thus, the expected value of A at the evolved state ρ_t is

$$\begin{aligned} \langle A \rangle_{\rho_t} &:= \text{Tr}[A\rho_t] = \text{Tr}[Ae^{-itH}\rho e^{itH}] \\ &= \text{Tr}[e^{itH}Ae^{-itH}\rho] = \text{Tr}[\tau_t(A)\rho] \\ &= \langle \tau_t(A) \rangle_\rho \end{aligned} \quad (2.5)$$

where

$$\tau_t(A) = e^{itH}Ae^{-itH} \quad (2.6)$$

is called the *Heisenberg* dynamics of the observable A .

2.3. Coupling quantum systems and entanglement

Consider two quantum systems acting on the Hilbert spaces \mathcal{H} and \mathcal{K} with Hamiltonians H_1 and H_2 , respectively. The “coupling” axiom of quantum mechanics says that the Hilbert space of the coupled system is the tensor product $\mathcal{H} \otimes \mathcal{K}$. The coupling Hamiltonian $H_0^{(2)}$ of the non-interacting systems is given by

$$H_0^{(2)} = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2. \quad (2.7)$$

If A and B are observables of the systems \mathcal{H} and \mathcal{K} , respectively, then they are extended as observables of $\mathcal{H} \otimes \mathcal{K}$ as

$$A \otimes \mathbb{1}, \quad \text{and} \quad \mathbb{1} \otimes B.$$

Moreover, if the two subsystems are in states ϕ and ψ respectively, then the coupled system is the state

$$\phi \otimes \psi. \quad (2.8)$$

The corresponding density matrix is

$$\rho = |\phi \otimes \psi\rangle\langle\phi \otimes \psi| = |\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi|. \quad (2.9)$$

Again, depending on the context, states of the forms (2.8) or (2.9) are called *product* states. A pure state is said to be *entangled* if it cannot be written as a product state. Note that this definition of “entangled states” does not apply for mixed states. We will not discuss it here because we are going to consider entanglement of pure states only. It is important to keep in mind that not all pure states in $\mathcal{H} \otimes \mathcal{K}$ are product states. The following theorem gives the general form of a state ϕ in $\mathcal{H} \otimes \mathcal{K}$.

THEOREM 2.1. (*Schmidt Decomposition*) *For $\phi \in \mathcal{H} \otimes \mathcal{K}$ with $\|\phi\| = 1$, there exist ONBs $\{e_j\}$ and $\{f_k\}$ of \mathcal{H} and \mathcal{K} , respectively, and a unique sequence $s_1 \geq s_2 \geq \dots \geq 0$ with $\sum_j s_j^2 = 1$ such that*

$$\phi = \sum_j s_j (e_j \otimes f_j). \quad (2.10)$$

In particular, ϕ is a product state (not entangled) if and only if $s_1 = 1$, $s_j = 0$ for $j \geq 2$.

PROOF. $\{e_j \otimes f_j\}_{j=1}^\infty$ is an orthonormal system, thus $\sum_j s_j^2 = 1$ follows from $\|\phi\| = 1$ once (2.10) is proven.

Pick ONBs $\{\tilde{e}_j\}$, $\{\tilde{f}_k\}$ in \mathcal{H} and \mathcal{K} , respectively. Let $\Phi = (\phi_{jk})$, be the matrix operator in $\ell^2(\mathbb{N})$ with $\phi_{jk} = \langle \tilde{e}_j \otimes \tilde{f}_k, \phi \rangle$, then

$$\sum_{j,k} |\phi_{jk}|^2 = \sum_{j,k} |\langle \tilde{e}_j \otimes \tilde{f}_k, \phi \rangle|^2 = \|\phi\|^2 = 1,$$

so $\Phi \in \mathcal{B}_2(\ell^2(\mathbb{N}))$, the Hilbert-Schmidt operators, in particular Φ is bounded and compact. The SVD of Φ gives sequences of ONBs $\{g_j\}$ and $\{h_k\}$ in $\ell^2(\mathbb{N})$ and a

sequence $s_1 \geq s_2 \geq \dots \geq 0$ such that

$$\Phi\psi = \sum_j s_j \langle g_j, \psi \rangle h_j$$

for every $\psi \in \ell^2(\mathbb{N})$. This can be rewritten as $\Phi = UDV^*$, where V is the unitary matrix operator with columns g_j , U is the unitary matrix operator with columns h_j , and $D = \text{diag}\{s_j, j \in \mathbb{N}\}$. Now, change the ONBs in \mathcal{H} and \mathcal{K} ,

$$e_j = \sum_{\ell} U_{\ell j} \tilde{e}_{\ell} \quad \text{and} \quad f_k = \sum_m \overline{V_{mk}} \tilde{f}_m \quad \forall j, k.$$

Using the new bases, we have

$$\langle e_j \otimes f_k, \phi \rangle = \sum_{\ell, m} \overline{U_{\ell j}} V_{mk} \langle \tilde{e}_{\ell} \otimes \tilde{f}_m, \phi \rangle \quad (2.11)$$

and

$$\langle \tilde{e}_{\ell} \otimes \tilde{f}_m, \phi \rangle = \phi_{\ell m} = \sum_{r, s} U_{lr} \underbrace{D_{rs}}_{s_r \delta_{rs}} \overline{V_{ms}} = \sum_r s_r U_{lr} \overline{V_{mr}}.$$

By substituting in equation (2.11), we get

$$\begin{aligned} \langle e_j \otimes f_k, \phi \rangle &= \sum_r s_r \underbrace{\sum_{\ell} \overline{U_{\ell j}} U_{lr}}_{\delta_{jr}} \underbrace{\sum_m V_{mk} \overline{V_{mr}}}_{\delta_{kr}} \\ &= \begin{cases} 0, & \text{if } j \neq k; \\ s_j, & \text{if } j = k. \end{cases} \end{aligned} \quad (2.12)$$

So, $\phi = \sum_j s_j (e_j \otimes f_j)$. □

A *Many-body system* is regarded as a generalization of the previous argument from the coupling of two systems to the coupling of n subsystems over the Hilbert spaces \mathcal{H}_i for $i \in \{1, 2, \dots, n\}$. In this case, the Hamiltonian $H_0^{(n)}$ is a generalization of (2.7) and it is acting over the Hilbert space

$$\mathcal{H}^{(n)} = \bigotimes_{j=1}^n \mathcal{H}_j.$$

For interacting many-body systems, we have a Hamiltonian of the form

$$H^{(n)} = H_0^{(n)} + W$$

where W is a self-adjoint operator on \mathcal{H}_n describing the interaction between the subsystems.

2.4. Entropy

Let ρ be a mixed state. That is, ρ is a non-negative self-adjoint operator with trace 1. This means that $\sigma(\rho) \subset [0, 1]$ and

$$\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$$

with $0 \leq \lambda_k \leq 1$ and $\sum_k \lambda_k = 1$.

Now, define

$$g(x) := \begin{cases} -x \log x, & x \in (0, 1] \\ 0, & x = 0 \end{cases}. \quad (2.13)$$

Here and throughout this text, we use \log to denote the natural logarithm, as opposed to the dyadic logarithm used in this context in the information theory literature. The distinction is irrelevant for our work, as we will not keep track of universal constants.

g is a non-negative, continuous function on $[0, 1]$ and has a maximum at $(\frac{1}{e}, \frac{1}{e})$.

The *von Neumann entropy* of ρ is given by

$$\mathcal{S}(\rho) := \text{Tr}[g(\rho)].$$

We note here that $g(\rho)$ is defined in the sense of the functional calculus and that, as $g \geq 0$, we have $g(\rho) \geq 0$, thus

$$\mathcal{S}(\rho) \geq 0. \quad (2.14)$$

We can also, slightly informally, define the von Neumann entropy as

$$\mathcal{S}(\rho) = -\text{Tr}[\rho \log \rho] = -\sum_k \lambda_k \log \lambda_k = \sum_k g(\lambda_k),$$

where one can see that $g(\rho) \in \mathcal{B}_1(\mathcal{H})$ if and only if $\mathcal{S}(\rho) < \infty$.

Here we list some properties of the entropy $\mathcal{S}(\rho)$,

LEMMA 2.1. (a) $\mathcal{S}(\rho) = 0$ if and only if ρ is a pure state.

(b) If $n = \dim \mathcal{H} < \infty$, then

$$\max_{\rho} \mathcal{S}(\rho) = \log n,$$

which is attained if and only if $\rho = \frac{1}{n} \mathbb{1}$.

(c) If $\dim \mathcal{H} = \infty$, then there exist mixed states ρ such that $\mathcal{S}(\rho) = \infty$.

PROOF. (a) This can be seen if we note that

$$\begin{aligned} \mathcal{S}(\rho) = 0 &\Leftrightarrow g(\lambda_k) = 0 \quad \forall k \\ &\Leftrightarrow \lambda_k \in \{0, 1\} \quad \forall k \\ &\Leftrightarrow \lambda_1 = 1, \lambda_k = 0 \quad k \geq 2 \\ &\Leftrightarrow \rho = |\phi_1\rangle\langle\phi_1|, \end{aligned}$$

where g is defined in (2.13).

(b) Let

$$\rho = \sum_{j=1}^n \lambda_j |\phi_j\rangle\langle\phi_j|.$$

Define the probability measure μ on $[0, 1]$ as

$$\mu := \frac{1}{n} \sum_{j=1}^n \delta_{\lambda_j}.$$

If we let $f(x) = x$ on $[0, 1]$, then since $-g$ is strictly convex on $[0, 1]$, we can apply

Jensen's inequality. We then have

$$\underbrace{-g\left(\int_{[0,1]} f d\mu\right)}_{=\frac{1}{n} \sum_{j=1}^n \lambda_j = \frac{1}{n}} \leq \underbrace{-\int_{[0,1]} g \circ f d\mu}_{=\frac{1}{n} \sum_{j=1}^n g(\lambda_j)}.$$

Thus, we have

$$\mathcal{S}(\rho) = \sum_{j=1}^n g(\lambda_j) \leq ng \left(\frac{1}{n} \right) = -n \frac{1}{n} \log \left(\frac{1}{n} \right) = \log n.$$

Furthermore, since $-g$ is strictly convex, we have equality if and only if $f(x)$ is constant for μ almost everywhere. This occurs if and only if $\text{supp } \mu$ is a singleton, i.e. $\lambda_1 = \dots = \lambda_n$. This means that $\lambda_j = 1/n$ for all j and $\rho = \frac{1}{n} \mathbb{1}$.

(d) In order to construct an example you need a sequence of eigenvalues which converge to zero sufficiently slow. The sequence

$$\lambda_k = \frac{C}{k(\log k)^2}$$

for $k \geq 2$ is one such sequence. The C is a normalization constant, guaranteeing $\sum_k \lambda_k = 1$, while one finds $-\sum_k \lambda_k \log \lambda_k = \infty$. \square

The next Lemma deals with the entropy of product states.

LEMMA 2.2. *Let ρ_1 and ρ_2 be two states in $\mathcal{B}_1(\mathcal{H})$ and $\mathcal{B}_1(\mathcal{K})$, respectively, then*

$$\mathcal{S}(\rho_1 \otimes \rho_2) = \mathcal{S}(\rho_1) + \mathcal{S}(\rho_2). \quad (2.15)$$

PROOF. Let $\{\lambda_j\}_j$, and $\{\mu_k\}_k$ be the eigenvalues of ρ_1 and ρ_2 , respectively. Note that the eigenvalues of $\rho_1 \otimes \rho_2$ are $\{\lambda_j \mu_k\}_{j,k}$. First, if one of the states say ρ_1 is a pure state, then it has eigenvalues 1 (with multiplicity one) and zeros. Thus, the nonzero eigenvalues of $\rho_1 \otimes \rho_2$ are the same as those of ρ_2 , which leads to (2.15). So, we will assume that $0 < \lambda_j, \mu_k < 1$ for all j and k .

$$\begin{aligned} \mathcal{S}(\rho_1 \otimes \rho_2) &= - \sum_{j,k} (\lambda_j \mu_k \log(\lambda_j \mu_k)) \\ &= - \sum_{j,k} (\lambda_j \mu_k \log \lambda_j) - \sum_{j,k} (\lambda_j \mu_k \log \mu_k) \\ &= - \sum_j (\lambda_j \log \lambda_j) - \sum_k (\mu_k \log \mu_k) \\ &= \mathcal{S}(\rho_1) + \mathcal{S}(\rho_2), \end{aligned} \quad (2.16)$$

where we used the fact that $\sum_j \lambda_j = \sum_k \mu_k = 1$. □

2.5. Partial traces

We present the definition of the partial trace operator as a part of the following theorem, see Chapter 2 in [Att].

THEOREM 2.2. [Att] *Let $T \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$. Then there exists a unique trace class operator $\text{Tr}_{\mathcal{K}} T \in \mathcal{B}_1(\mathcal{H})$ such that*

$$\langle f, (\text{Tr}_{\mathcal{K}} T)h \rangle = \sum_n \langle f \otimes g_n, T (h \otimes g_n) \rangle \quad (2.17)$$

for all $f, h \in \mathcal{H}$ and every ONB $\{g_n\}$ in \mathcal{K} .

We call $\text{Tr}_{\mathcal{K}} T$ the *partial trace* of T with respect to \mathcal{K} or the *reduced state* of T to \mathcal{H} . Similarly, one can define $\text{Tr}_{\mathcal{H}} T$ by

$$\langle \phi, (\text{Tr}_{\mathcal{H}} T)\psi \rangle = \sum_n \langle e_n \otimes \phi, T (e_n \otimes \psi) \rangle$$

for all $\phi, \psi \in \mathcal{K}$ and every ONB $\{e_n\}$ in \mathcal{H} .

We list some properties of the partial trace in the following lemma.

LEMMA 2.3. (a) *If $T \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$, then*

$$\text{Tr}[T] = \text{tr}[\text{Tr}_{\mathcal{K}} T] = \text{tr}[\text{Tr}_{\mathcal{H}} T].$$

(b) *If $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{B}_1(\mathcal{K})$ then $A \otimes B \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$ and*

$$\text{Tr}_{\mathcal{K}}(A \otimes B) = \text{tr}[B] A, \quad \text{Tr}_{\mathcal{H}}(A \otimes B) = \text{tr}[A] B. \quad (2.18)$$

(c) *If $A \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$ then*

$$\text{Tr}_{\mathcal{K}}(A \otimes \mathbb{1})T = A \text{Tr}_{\mathcal{K}} T. \quad (2.19)$$

Note that here and throughout the thesis, we use the two notations Tr and tr for the trace. The former is to denote the trace over the full Hilbert space $\mathcal{H} \otimes \mathcal{K}$, and the second with a little “t” is to denote the trace over the smaller Hilbert spaces \mathcal{H} or \mathcal{K} .

PROOF. (a) We prove that

$$\text{tr} [\text{Tr}_{\mathcal{K}} T] = \text{Tr}[T].$$

Proving the other equality that the partial trace with respect to \mathcal{H} has the same trace as T is identical. Let $\{e_j\}$ and $\{g_\ell\}$ be ONBs of \mathcal{H} and \mathcal{K} , respectively.

$$\begin{aligned} \text{tr} [\text{Tr}_{\mathcal{K}} T] &= \sum_j \langle e_j, (\text{Tr}_{\mathcal{K}} T)e_j \rangle & (2.20) \\ &= \sum_{j,\ell} \langle e_j \otimes g_\ell, T(e_j \otimes g_\ell) \rangle \\ &= \text{Tr}[T]. \end{aligned}$$

In the last step we used that $\{e_j \otimes g_\ell\}_{j,\ell}$ is an ONB of $\mathcal{H} \otimes \mathcal{K}$.

(b) Since A and B are trace class operators, then using the SVD there are non-negative sequences $\{s_j\}$ and $\{r_j\}$ and ONBs $\{f_j\}$, $\{g_j\}$ of \mathcal{H} and $\{\tilde{f}_j\}$ and $\{\tilde{g}_j\}$ of \mathcal{K} such that

$$A = \sum_j s_j |f_j\rangle\langle g_j| \quad \text{and} \quad B = \sum_j r_j |\tilde{f}_j\rangle\langle \tilde{g}_j| \quad (2.21)$$

with

$$\|A\|_1 = \sum_j s_j \quad \text{and} \quad \|B\|_1 = \sum_j r_j. \quad (2.22)$$

Then the tensor product gives

$$\begin{aligned} A \otimes B &= \sum_{j,k} s_j r_k |f_j\rangle\langle g_j| \otimes |\tilde{f}_k\rangle\langle \tilde{g}_k| & (2.23) \\ &= \sum_{j,k} s_j r_k |f_j \otimes \tilde{f}_k\rangle\langle g_j \otimes \tilde{g}_k| \end{aligned}$$

which yields that

$$\|A \otimes B\|_1 = \sum_{j,k} s_j r_k = \|A\|_1 \|B\|_1 < \infty.$$

We prove the first statement in (2.18), the proof of the second equality is identical.

For any $f, h \in \mathcal{H}$ and ONB $\{g_\ell\}$ of \mathcal{K} .

$$\begin{aligned} \langle f, (\text{Tr}_{\mathcal{K}}(A \otimes B)) h \rangle &= \sum_{\ell} \langle f \otimes g_{\ell}, (A \otimes B) h \otimes g_{\ell} \rangle \\ &= \sum_{\ell} \langle f, Ah \rangle \langle g_{\ell}, Bg_{\ell} \rangle \\ &= \langle f, \text{tr}[B]A h \rangle. \end{aligned} \quad (2.24)$$

(c) First note that $(A \otimes \mathbb{1}) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ and that $(A \otimes \mathbb{1})T \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$, see for example Theorem 7.8 in [Wei80]. For any $f, h \in \mathcal{H}$ and ONB $\{g_n\}$ of \mathcal{K} :

$$\begin{aligned} \langle f, (\text{Tr}_{\mathcal{K}}(A \otimes \mathbb{1})T) h \rangle &= \sum_n \langle f \otimes g_n, ((A \otimes \mathbb{1})T) h \otimes g_n \rangle \\ &= \sum_n \langle (A^* f) \otimes g_n, T(h \otimes g_n) \rangle \\ &= \langle A^* f, (\text{Tr}_{\mathcal{K}} T) h \rangle = \langle f, (A \text{Tr}_{\mathcal{K}} T) h \rangle. \end{aligned} \quad (2.25)$$

This completes the proof. □

Our interest will be in states, here we prove that reduced states of states are states as well.

LEMMA 2.4. *If $\rho \in \mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$ is a state then $\text{Tr}_{\mathcal{K}} \rho$ is a state (≥ 0 and has trace one).*

PROOF. For any $f, h \in \mathcal{H}$ and ONB $\{g_\ell\}$ of \mathcal{K} , we have

$$\begin{aligned} \langle f, (\text{Tr}_{\mathcal{K}} \rho) h \rangle &= \sum_{\ell} \langle f \otimes g_{\ell}, \rho(h \otimes g_{\ell}) \rangle \\ &= \sum_{\ell} \langle \rho(f \otimes g_{\ell}), h \otimes g_{\ell} \rangle \\ &= \overline{\sum_{\ell} \langle h \otimes g_{\ell}, \rho(f \otimes g_{\ell}) \rangle} \end{aligned} \quad (2.26)$$

$$\begin{aligned}
&= \overline{\langle h, (\text{Tr}_{\mathcal{K}} \rho) f \rangle} \\
&= \langle (\text{Tr}_{\mathcal{K}} \rho) f, h \rangle.
\end{aligned}$$

Since ρ is closed, then

$$(\text{Tr}_{\mathcal{K}} \rho)^* = \text{Tr}_{\mathcal{K}} \rho.$$

The following simple argument shows the non-negativity of the reduced state

$$\langle f, (\text{Tr}_{\mathcal{K}} \rho) f \rangle = \sum_{\ell} \langle f \otimes g_{\ell}, \rho(f \otimes g_{\ell}) \rangle. \quad (2.27)$$

The left hand side is a sum of non-negative diagonal elements of ρ , thus $\text{Tr}_{\mathcal{K}} \rho \geq 0$.

The fact that the reduced state has trace one follows directly from Lemma 2.3. \square

The following Lemma establishes links between reduced states and entanglements.

LEMMA 2.5. *If ρ is a pure state in $\mathcal{B}_1(\mathcal{H} \otimes \mathcal{K})$, then the reduced states $\text{Tr}_{\mathcal{K}} \rho$ and $\text{Tr}_{\mathcal{H}} \rho$ are pure states if and only if ρ is a product state (not entangled).*

PROOF. Let $\phi \in \mathcal{H} \otimes \mathcal{K}$ with $\|\phi\| = 1$, using the Schmidt decomposition, there exist orthonormal systems $\{e_j\}$ and $\{f_j\}$ of \mathcal{H} and \mathcal{K} , respectively, and a non-increasing non-negative sequence $\{s_j\}$ such that

$$\phi = \sum_j s_j e_j \otimes f_j, \quad \text{with} \quad \sum_j s_j^2 = 1.$$

Then

$$\begin{aligned}
\rho = |\phi\rangle\langle\phi| &= \sum_{j,k} s_j s_k |e_j \otimes f_j\rangle\langle e_k \otimes f_k| \\
&= \sum_{j,k} s_j s_k |e_j\rangle\langle e_k| \otimes |f_j\rangle\langle f_k|.
\end{aligned}$$

By taking the partial trace with respect to \mathcal{K} ,

$$\text{Tr}_{\mathcal{K}} \rho = \sum_{j,k} s_j s_k \text{Tr}_{\mathcal{K}} (|e_j\rangle\langle e_k| \otimes |f_j\rangle\langle f_k|) \quad (2.28)$$

$$\begin{aligned}
&= \sum_{j,k} s_j s_k |e_j\rangle\langle e_k| \delta_{jk} \\
&= \sum_j s_j^2 |e_j\rangle\langle e_j|.
\end{aligned}$$

In the second step we used the property (a) in Lemma 2.18 noting that $\text{tr}[|f_j\rangle\langle f_k|] = \delta_{jk}$.

Similarly, we can see that

$$\text{Tr}_{\mathcal{H}} \rho = \sum_j s_j^2 |f_j\rangle\langle f_j|.$$

This means that the reduced states are pure states if and only if $s_1 = 1$ and $s_k = 0$ for $k > 1$. By substituting in (2.28) we get that

$$\rho = |e_1\rangle\langle e_1| \otimes |f_1\rangle\langle f_1|$$

which is a product state. □

2.6. Entanglement and von Neumann entanglement entropy

We now define the *entanglement entropy* as part of the following theorem.

THEOREM 2.3 (Entanglement Entropy). *Let ρ be a pure state in $\mathcal{H} \otimes \mathcal{K}$. Then its (bipartite) entanglement entropy is defined as*

$$\mathcal{E}(\rho) = \mathcal{S}(\text{Tr}_{\mathcal{H}} \rho) = \mathcal{S}(\text{Tr}_{\mathcal{K}} \rho), \quad (2.29)$$

where $\mathcal{S}(\cdot)$ denotes the von Neumann entropy in \mathcal{H} and \mathcal{K} , respectively.

What makes the above a theorem is that it needs to be proven that the two different expression given for the entanglement entropy are the same.

PROOF OF THEOREM 2.3. Let $\varphi \in \mathcal{H} \otimes \mathcal{K}$ with $\|\varphi\| = 1$ and let $\varphi = \sum_j s_j e_j \otimes f_j$ be its Schmidt decomposition as given by Proposition 2.1. Then

$$\rho = |\varphi\rangle\langle\varphi| = \sum_{j,\ell} s_j s_\ell |e_j \otimes f_j\rangle\langle e_\ell \otimes f_\ell|.$$

Then using the same calculation as in the proof of Lemma 2.5, we get

$$\mathrm{Tr}_{\mathcal{K}} \rho = \sum_j s_j^2 |e_j\rangle\langle e_j|, \quad \text{and} \quad \mathrm{Tr}_{\mathcal{H}} \rho = \sum_j s_j^2 |f_j\rangle\langle f_j|.$$

i.e. in the ONBs $\{e_r\}$ and $\{f_r\}$, respectively, given by the Schmidt decomposition, the two partial traces of ρ have the same diagonal matrix representation. In particular,

$$\mathcal{S}(\mathrm{Tr}_{\mathcal{H}} \rho) = \mathcal{S}(\mathrm{Tr}_{\mathcal{K}} \rho) = - \sum_r s_r^2 \log s_r^2. \quad (2.30)$$

□

The next Lemma explains why the entanglement entropy is considered as one of the best tools to quantify entanglement of pure states.

LEMMA 2.6. (a) $\mathcal{E}(\rho) \geq 0$.

(b) $\mathcal{E}(\rho) = 0$ if and only if ρ is a product state.

PROOF. (a) is a direct consequence of (2.14).

(b) Using the definition of the entanglement entropy, $\mathcal{E}(\rho) = 0$ if and only if $\mathcal{S}(\mathrm{Tr}_{\mathcal{H}} \rho) = \mathcal{S}(\mathrm{Tr}_{\mathcal{K}} \rho) = 0$ which is satisfied if and only if (Lemma 2.1) $\mathrm{Tr}_{\mathcal{H}} \rho$ and $\mathrm{Tr}_{\mathcal{K}} \rho$ are pure states. Lemma 2.5 proves that the latter is correct if and only if ρ is a product state. □

CHAPTER 3

The Theory of Finite Fermionic Systems

In this chapter we include some background from the theory of finite Fermionic systems, which will be used in Chapter 5. Some of this is well known in theoretical physics and can also be found in the mathematical physics literature, e.g. [BR97] or [BLS94]. We include a self-contained presentation of this material here.

3.1. Fermionic operators

In this section we define the Fermionic operators/systems and list some of their important properties. Then we present the Jordan-Wigner Fermionic systems, and finally, we study the Bogoliubov transformation.

In a Hilbert space \mathcal{H} of dimension $\dim \mathcal{H} = 2^n$, we call

$$\mathcal{D} = (d_1, d_1^*, d_2, d_2^*, \dots, d_n, d_n^*)^t \quad (3.1)$$

a *Fermionic system* if the operators $d_j \in \mathcal{B}(\mathcal{H})$ and their adjoints satisfy the *Canonical Anticommutation Relations* (CAR)

$$\{d_j, d_k^*\} = \delta_{jk} \mathbb{1}, \quad \{d_j, d_k\} = \{d_j^*, d_k^*\} = 0 \quad \text{for all } j, k = 1, \dots, n, \quad (3.2)$$

where we use $\{\cdot, \cdot\} : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ to denote the *Anti-commutator*. i.e. For A and B in $\mathcal{B}(\mathcal{H})$,

$$\{A, B\} := AB + BA. \quad (3.3)$$

We will use later $[\cdot, \cdot]$ to denote the *Commutator*, i.e.

$$[A, B] := AB - BA. \quad (3.4)$$

Note that the statements in (3.2) are equivalent to

$$\mathcal{D}\mathcal{D}^* + J(\mathcal{D}\mathcal{D}^*)^t J = \mathbb{1}_{2n}, \text{ where } J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n}. \quad (3.5)$$

Here we use the notation $(\cdot)^{\oplus n}$ to denote the n -fold direct sum corresponding to the natural order of canonical basis of \mathbb{C}^{2n} , i.e, J is the block diagonal matrix with blocks

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Next we state some basic properties of Fermionic systems.

LEMMA 3.1. *If \mathcal{D} is a Fermionic system in \mathcal{H} , then the operators $\{d_j, d_j^*, j = 1, 2, \dots, n\}$ are linearly independent in $\mathcal{B}(\mathcal{H})$.*

PROOF. Let $\{\alpha_j\}_{j=1}^n$ and $\{\beta_j\}_{j=1}^n$ in \mathbb{C} , such that

$$A := \sum_{j=1}^n (\alpha_j d_j + \beta_j d_j^*) = 0. \quad (3.6)$$

Now, for $k = 1, 2, \dots, n$ and using the linearity of $\{\cdot, \cdot\}$,

$$\{A, d_k\} = 0 \Rightarrow \beta_k \{d_k^*, d_k\} = 0 \Rightarrow \beta_k = 0 \quad (3.7)$$

$$\{A, d_k^*\} = 0 \Rightarrow \alpha_k \{d_k, d_k^*\} = 0 \Rightarrow \alpha_k = 0, \quad (3.8)$$

and this completes the proof. \square

In the following, we list some important algebraic properties of systems of Fermionic operators. For a somewhat useful introduction of these properties see [Nie05]. For a brief treatment see Proposition II.6.2 in [Sim93]. First, the operators $d_j^* d_j$, $j = 1, \dots, n$, are pairwise commuting orthogonal projections. This can be checked easily using the CAR of the d_j operators. Thus, they can be simultaneously diagonalizable with eigenvalues 1's and 0's. In other words, there exists a unit vector ψ such that for all j

$$d_j^* d_j \psi = \gamma_j \psi \quad \text{with } \gamma_j \in \{0, 1\}. \quad (3.9)$$

Second, note that if ψ is a unit eigenvector of $d_j^*d_j$ with eigenvalue 1, then $d_j\psi$ is a unit eigenvector of $d_j^*d_j$ with eigenvalue 0, for this reason, d_j is called *annihilation operator*. Maybe, the only non-obvious fact in the past statement is that $d_j\psi$ is a unit vector, this follows from the following argument:

$$\|d_j\psi\|^2 = \langle d_j\psi, d_j\psi \rangle = \langle \psi, d_j^*d_j\psi \rangle = \langle \psi, \psi \rangle = 1. \quad (3.10)$$

Similarly, if ϕ is a unit eigenvector of $d_j^*d_j$ with eigenvalue 0, then by applying the *creation operator* d_j^* on ϕ we get is a unit eigenvector $d_j^*\phi$ of $d_j^*d_j$ with eigenvalue 1. Back to (3.9), and define

$$\Omega := (d_1)^{\gamma_1}(d_2)^{\gamma_2} \dots (d_n)^{\gamma_n}\psi. \quad (3.11)$$

It follows directly from an iterative application of (3.10) that $\|\Omega\| = 1$ and for all j

$$d_j^*d_j\Omega = 0, \text{ thus } \Omega \in \bigcap_{j=1}^n \ker(d_j^*d_j).$$

For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^n$ use successive d_j^* to define

$$\psi_\alpha := (d_1^*)^{\alpha_1} \dots (d_n^*)^{\alpha_n}\Omega, \quad (3.12)$$

$\{\psi_\alpha, \alpha \in \{0, 1\}^n\}$ form an orthonormal system of common eigenvectors for the $d_j^*d_j$:

$$d_j^*d_j\psi_\alpha = \begin{cases} 0, & \text{if } \alpha_j = 0, \\ \psi_\alpha, & \text{if } \alpha_j = 1. \end{cases} \quad (3.13)$$

In the given case $\dim \mathcal{H} = 2^n$, and thus the vectors ψ_α form an ONB. In particular, the normalized vector Ω is unique up to a trivial phase. It is frequently referred to as the *vacuum state*.

LEMMA 3.2. *Let \mathcal{H} and $\tilde{\mathcal{H}}$ be 2^n dimensional Hilbert spaces and*

$$\mathcal{D} = (d_1, d_1^*, \dots, d_n, d_n^*)^t$$

be a Fermionic system in \mathcal{H} . Then

$$\tilde{\mathcal{D}} = (\tilde{d}_1, \tilde{d}_1^*, \dots, \tilde{d}_n, \tilde{d}_n^*)^t \quad (3.14)$$

is a Fermionic system in $\tilde{\mathcal{H}}$ if and only if there exists a unitary operator $U : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ such that $U^* \tilde{d}_j U = d_j$ for all $1 \leq j \leq n$. The unitary operator U is characterized by $U \phi_\alpha = \tilde{\phi}_\alpha$ for all $\alpha \in \{0, 1\}^n$, where $\{\phi_\alpha\}$ and $\{\tilde{\phi}_\alpha\}$ are the ONBs associated with \mathcal{D} and $\tilde{\mathcal{D}}$ through (3.12).

PROOF. Let $\tilde{\Omega}$ and Ω be the vacuum states of $\tilde{\mathcal{D}}$ and \mathcal{D} , respectively. For the basis vectors $\tilde{\phi}_\alpha = (\tilde{d}_1^*)^{\alpha_1} \dots (\tilde{d}_n^*)^{\alpha_n} \tilde{\Omega}$ and $\phi_\alpha = (d_1^*)^{\alpha_1} \dots (d_n^*)^{\alpha_n} \Omega$, let $U : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ the unitary determined by $U \phi_\alpha = \tilde{\phi}_\alpha$ for all α . Then for any j we have

$$\begin{aligned} \tilde{d}_j^* U \phi_\alpha &= \tilde{d}_j^* \tilde{\phi}_\alpha \\ &= \begin{cases} 0, & \text{if } \alpha_j = 1; \\ (-1)^{\sum_{k=1}^j \alpha_k} \tilde{\phi}_{\alpha+e_j} = (-1)^{\sum_{k=1}^j \alpha_k} U \phi_{\alpha+e_j}, & \text{if } \alpha_j = 0. \end{cases} \\ &= U d_j^* \phi_\alpha, \end{aligned} \quad (3.15)$$

here e_j is the j -th canonical basis vector of \mathbb{C}^n . Thus $d_j^* = U^* \tilde{d}_j^* U$ and $U^* \tilde{d}_j U = d_j$ for all $1 \leq j \leq n$. The converse follows easily using

$$\{\tilde{d}_j^{\#_1}, \tilde{d}_k^{\#_2}\} = U \{d_j^{\#_1}, d_k^{\#_2}\} U^* \quad \text{for any } 1 \leq j, k \leq n, \quad (3.16)$$

where $\#_1$ and $\#_2$ stand for $*$ or nothing. \square

LEMMA 3.3. Let \mathcal{D} be a Fermionic system in \mathcal{H} and \mathcal{A} be the \star -algebra generated by the components of \mathcal{D} . Then $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

PROOF. We need to show that for any $\alpha, \beta \in \{0, 1\}^n$ there exists an operator $A_{\alpha, \beta} \in \mathcal{A}$, such that $A_{\alpha, \beta} \psi_\alpha = \psi_\beta$ and $A_{\alpha, \beta} \psi_{\tilde{\alpha}} = 0$ for $\alpha \neq \tilde{\alpha}$. By defining

$$A_{\alpha, \beta} := \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \left(\prod_{j=1}^n d_j^{\alpha_j} \right), \quad (3.17)$$

and note that here and throughout this thesis operator products as on the right hand side of (3.17) are to be read from right to left, i.e. $\prod_{j=1}^m d_j = d_m \dots d_1$. The following calculation shows that each basis element is mapped to a basis element,

$$\begin{aligned}
A_{\alpha,\beta}\psi_\alpha &= \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \left(\prod_{j=1}^n (d_j)^{\alpha_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\alpha_{n-j}} \right) \Omega \\
&= \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \\
&\quad \left((d_n)^{\alpha_n} (d_{n-1})^{\alpha_{n-1}} \dots (d_1)^{\alpha_1} (d_1^*)^{\alpha_1} (d_2^*)^{\alpha_2} \dots (d_n^*)^{\alpha_n} \Omega \right) \\
&= \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \Omega = \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \psi_\beta = \psi_\beta
\end{aligned}$$

where we used $d_j d_j^* d_k \Omega = d_k \Omega$ for $j \neq k$ in the last step. Finally, we will show that $A_{\alpha,\beta}\psi_{\tilde{\alpha}} = 0$ for $\alpha \neq \tilde{\alpha}$, we have

$$A_{\alpha,\beta}\psi_{\tilde{\alpha}} = \left(\prod_{j=1}^n (d_j d_j^*)^{1-\beta_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\beta_{n-j}} \right) \left(\prod_{j=1}^n (d_j)^{\alpha_j} \right) \left(\prod_{j=0}^{n-1} (d_{n-j}^*)^{\tilde{\alpha}_{n-j}} \right) \Omega. \tag{3.18}$$

Since $\tilde{\alpha} \neq \alpha$, there exists j_0 such that either $(\alpha_{j_0}, \tilde{\alpha}_{j_0}) = (1, 0)$ or $(\alpha_{j_0}, \tilde{\alpha}_{j_0}) = (0, 1)$. In the first case d_{j_0} will commute (up to signs) with all the terms to its right and thus annihilate Ω . In the second case $d_{j_0}^*$ will appear twice in (3.18) and commutes (again up to signs) with all the terms in between its two appearances. As $(d_{j_0}^*)^2 = 0$, we again get $A_{\alpha,\beta}\psi_{\tilde{\alpha}} = 0$. \square

Let $f, g : [1, n] \rightarrow \mathbb{C}$ and $\{d_j\}_{j=1}^n$ be a set of Fermionic operators, then define

$$\begin{aligned}
d(f) &:= \sum_{j=1}^n \bar{f}_j d_j, \text{ where } f := \begin{bmatrix} f_1 & f_2 & \dots & f_n \end{bmatrix}^t, f_j := f(j). \\
d^*(g) &:= \sum_{k=1}^n g_k d_k^*, \text{ where } g := \begin{bmatrix} g_1 & g_2 & \dots & g_n \end{bmatrix}^t, g_k := g(k).
\end{aligned} \tag{3.19}$$

For more general form of combinations, we define

$$D(f, g) := d(f) + d^*(g). \tag{3.20}$$

The following Corollary follows directly from Lemma 3.3

COROLLARY 3.1. *Finite products of $D(f_j, g_j)$ generate the \star -algebra $\mathcal{A} = \mathcal{B}(\mathcal{H})$.*

LEMMA 3.4. *Let ρ and $\tilde{\rho}$ be operators on $\mathcal{B}(\mathcal{H})$. Assume that for all $m \in \mathbb{N}$, $f_j, g_k : [1, m] \rightarrow \mathbb{C}$ where $j, k = 1, 2, \dots, m$ we have*

$$\left\langle \prod_{j=1}^m D(f_j, g_j) \right\rangle_{\rho} = \left\langle \prod_{j=1}^m D(f_j, g_j) \right\rangle_{\tilde{\rho}} \quad (3.21)$$

then $\rho = \tilde{\rho}$.

PROOF. Let us consider any ONB $\{\phi_{\alpha}\}$ of \mathcal{H} , then using Lemma 3.1, for all $A \in \mathcal{B}(\mathcal{H})$ we have

$$\langle A \rangle_{\rho} = \langle A \rangle_{\tilde{\rho}} \Rightarrow \text{Tr}[A\rho] = \text{Tr}[A\tilde{\rho}], \text{ for all } A \in \mathcal{B}(\mathcal{H}). \quad (3.22)$$

Then for $A = A_{\alpha, \beta} := |\phi_{\alpha}\rangle\langle\phi_{\beta}|$, (3.22) reads

$$\langle\phi_{\beta}, \rho\phi_{\alpha}\rangle = \langle\phi_{\beta}, \tilde{\rho}\phi_{\alpha}\rangle \quad (3.23)$$

Thus, $\rho = \tilde{\rho}$ as claimed. \square

3.1.1. Jordan-Wigner Fermionic system. In this subsection we introduce a special important Fermionic operators on $\mathcal{B}(\mathcal{H})$, where $\mathcal{H} = \bigotimes_{j \in \Lambda} \mathbb{C}^2$, and $\Lambda = [1, n] := \{1, 2, \dots, n\}$. Let

$$\sigma_j^{\bullet} = \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes \sigma^{\bullet} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad \bullet \in \{x, y, z\}, \quad (3.24)$$

with non-trivial entries in the j -th component of the tensor product, and choose the standard representation

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.25)$$

of the Pauli matrices. We note that,

$$(\sigma^x)^2 = (\sigma^y)^2 = (\sigma^z)^2 = \mathbb{1}, \quad \{\sigma^x, \sigma^y\} = \{\sigma^x, \sigma^z\} = \{\sigma^y, \sigma^z\} = 0. \quad (3.26)$$

Define

$$a := \frac{1}{2}(\sigma^x - i\sigma^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (3.27)$$

Let us consider the canonical basis of \mathbb{C}^2 with the up-down spin notation:

$$e_\uparrow := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_\downarrow := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3.28)$$

It is observed that

$$ae_\uparrow = e_\downarrow \text{ and } a^*e_\downarrow = e_\uparrow,$$

and for this, a and a^* are called *lowering* and *raising* operators, respectively. Note that

$$a^*a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad aa^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (3.29)$$

are orthogonal projections. As above we write

$$a_j = \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}, \quad (3.30)$$

$$a_j^* = \mathbb{1} \otimes \dots \otimes \mathbb{1} \otimes a^* \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1}. \quad (3.31)$$

The raising and lowering operators satisfy the mixed commutator and anti-commutator relations

$$\{a_j, a_j^*\} = \mathbb{1}, \quad (a_j)^2 = (a_j^*)^2 = 0, \quad (3.32)$$

$$[a_j, a_k^*] = [a_j^*, a_k^*] = [a_j, a_k] = 0 \quad \text{for } j \neq k. \quad (3.33)$$

The creation and annihilation operators are defined as follows,

$$c_j := \sigma_1^z \dots \sigma_{j-1}^z a_j, \quad c_j^* = \sigma_1^z \dots \sigma_{j-1}^z a_j^*, \quad j = 1, \dots, n. \quad (3.34)$$

From this it follows readily that the c_j operators satisfy the CAR:

$$\{c_j, c_k^*\} = \delta_{jk} \mathbb{1}, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0 \quad \text{for all } j, k = 1, \dots, n. \quad (3.35)$$

This transformation from a set of Pauli operators to a set of Fermionic operators dates back at least to the paper of Jordan and Wigner [JW28] in 1928. We call

$$\mathcal{C} := (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t$$

the *Jordan-Wigner Fermionic system*.

We will often need to deal with the *Local Jordan-Wigner systems* that are defined over an interval $\Lambda_0 := \{r, r+1, \dots, r+\ell-1\} \subset \Lambda$ as follows, see also Figure 3.1.1:

$$c_j^{(1)} := (\sigma^z)^{\otimes(j-1)} \otimes a \otimes \mathbb{1}^{\otimes(\ell-j)} \quad j \in \Lambda_0. \quad (3.36)$$

It is easy to check that the $c_j^{(1)}$'s are Fermionic operators in $\mathcal{H}_1 := \bigotimes_{j \in \Lambda_0} \mathbb{C}^2$. We use \mathcal{C}_1 to denote Fermionic system associated with the $c_j^{(1)}$.

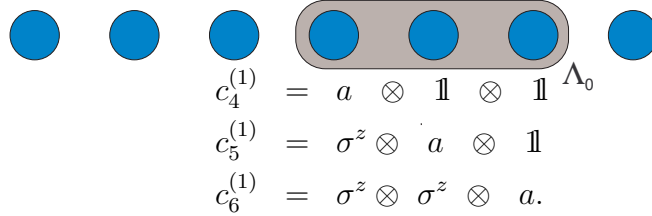


FIGURE 3.1. Local Jordan-Wigner operators when $n = 7$ and $\ell = 3$.

3.1.2. Bogoliubov transformation. In this subsection we define and discuss the Bogoliubov transformation between two Fermionic systems.

A matrix $W \in \mathbb{C}^{2n \times 2n}$ is called a Bogoliubov matrix if W is unitary and

$$WJW^t = J, \quad \text{where } J = (\sigma^x)^{\oplus n}. \quad (3.37)$$

The reason for using this terminology is that, for the finite Fermionic systems considered here, Bogoliubov matrices implement Bogoliubov transformations:

THEOREM 3.1. Let \mathcal{D} be a Fermionic system in \mathcal{H} and $W \in \mathbb{C}^{2n \times 2n}$. Then

$$\tilde{\mathcal{D}} := W\mathcal{D} \quad (3.38)$$

is a Fermionic system in \mathcal{H} if and only if W is a Bogoliubov matrix.

PROOF. We will assume first that W is a Bogoliubov matrix and \mathcal{D} is a Fermionic system, this means that W is unitary, $WJW^t = J$, and

$$\mathcal{D}\mathcal{D}^* + J(\mathcal{D}\mathcal{D}^*)^t J = \mathbb{1}_{2n}. \quad (3.39)$$

Condition (3.37) is equivalent to say that $J\bar{W} = WJ$ and $W^t J = JW^*$, then

$$\begin{aligned} \tilde{\mathcal{D}}\tilde{\mathcal{D}}^* + J(\tilde{\mathcal{D}}\tilde{\mathcal{D}}^*)^t J &= W\mathcal{D}\mathcal{D}^*W^* + J(W\mathcal{D}\mathcal{D}^*W^*)^t J \\ &= W\mathcal{D}\mathcal{D}^*W^* + J\bar{W}(\mathcal{D}\mathcal{D}^*)^t W^t J \\ &= W\mathcal{D}\mathcal{D}^*W^* + WJ(\mathcal{D}\mathcal{D}^*)^t JW^* \\ &= W(\mathcal{D}\mathcal{D}^* + J(\mathcal{D}\mathcal{D}^*)^t J)W^* \\ &= W\mathbb{1}_{2n}W^* = \mathbb{1}_{2n}. \end{aligned} \quad (3.40)$$

Thus, $\tilde{\mathcal{D}}$ is a Fermionic system. By performing a simple change of basis, the converse can be restated as: If \mathcal{D} and $\tilde{\mathcal{D}}$ are two Fermionic systems related by

$$\left(\tilde{d}_1, \tilde{d}_2, \dots, \tilde{d}_n, (\tilde{d}_1)^*, \dots, (\tilde{d}_n)^* \right)^t = \hat{W} (d_1, d_2, \dots, d_n, (d_1)^*, \dots, (d_n)^*)^t \quad (3.41)$$

then \hat{W} is unitary and satisfies

$$\hat{W}J_1\hat{W}^t = J_1, \text{ where } J_1 := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \quad (3.42)$$

To show this, we will show that \hat{W} is unitary with the following block structure

$$\hat{W} = \begin{bmatrix} K & L \\ \bar{L} & \bar{K} \end{bmatrix}. \quad (3.43)$$

First, write \hat{W} in the generic block form,

$$\hat{W} = \begin{bmatrix} K & L \\ M & N \end{bmatrix}. \quad (3.44)$$

From the transformation (3.41) we get that for $j = 1, 2, \dots, n$

$$\tilde{d}_j = \sum_{k=1}^n (K_{jk}d_k + L_{jk}d_k^*), \quad (3.45)$$

$$\tilde{d}_j^* = \sum_{k=1}^n (M_{jk}d_k + N_{jk}d_k^*). \quad (3.46)$$

By taking the adjoint of the left hand side of (3.45)

$$\tilde{d}_j^* = \sum_{k=1}^n (\overline{K_{jk}}d_k^* + \overline{L_{jk}}d_k). \quad (3.47)$$

Then comparing it with (3.46), using that the elements of a Fermionic system are linearly independent, see Lemma 3.1, we get

$$N_{jk} = \overline{K_{jk}}, \quad M_{jk} = \overline{L_{jk}}, \quad \text{for } k = 1, 2, \dots, n. \quad (3.48)$$

Since this is true also for all $j = 1, 2, \dots, n$, then this means that

$$M = \overline{L}, \quad \text{and } N = \overline{K}. \quad (3.49)$$

Next, we will prove that \hat{W} is a unitary matrix that is the rows are ONB of \mathbb{C}^{2n} .

Using (3.45), (3.46) and (3.49), we have the following

$$\begin{aligned} \{\tilde{d}_j, \tilde{d}_j^*\} = 1 &\Rightarrow \left\{ \sum_{m=1}^n (K_{jm}d_m + L_{jm}d_m^*), \sum_{l=1}^n (\overline{L_{jl}}d_l + \overline{K_{jl}}d_l^*) \right\} = 1 \quad (3.50) \\ &\Rightarrow \sum_{k,l=1}^n (K_{jk}\overline{K_{jl}}\{d_k, d_l^*\}) + \sum_{k,l=1}^n (L_{jk}\overline{L_{jl}}\{d_k^*, d_l\}) = 1 \\ &\Rightarrow \sum_{k=1}^n K_{jk}\overline{K_{jk}} + L_{jk}\overline{L_{jk}} = 1 \end{aligned}$$

$$\Rightarrow \sum_{z=1}^{2n} |\hat{W}_{jz}|^2 = 1.$$

This proves that the rows of \hat{W} are normalized. To prove that different rows of \hat{W} are orthogonal we use the following two arguments: First, using (3.45), (3.46) and (3.49), we have the following for $j \neq k$

$$\begin{aligned} \{\tilde{d}_j, \tilde{d}_k^*\} = 0 &\Rightarrow \left\{ \sum_{m=1}^n (K_{jm}d_m + L_{jm}d_m^*), \sum_{l=1}^n (\overline{L_{kl}}d_l + \overline{K_{kl}}d_l^*) \right\} = 0 \quad (3.51) \\ &\Rightarrow \sum_{m,l=1}^n (K_{jm}\overline{K_{kl}}\{d_m, d_l^*\}) + \sum_{m,l=1}^n (L_{jm}\overline{L_{kl}}\{d_m^*, d_l\}) = 0 \\ &\Rightarrow \sum_{m=1}^n K_{jm}\overline{K_{km}} + L_{jm}\overline{L_{km}} = 0 \\ &\Rightarrow \sum_{z=1}^{2n} \hat{W}_{jz}\overline{\hat{W}_{kz}} = 0 \text{ and } \sum_{z=1}^{2n} \hat{W}_{(j+n)z}\overline{\hat{W}_{(j+n)z}} = 0. \end{aligned}$$

Second, for $j \neq k$, we have

$$\begin{aligned} \{\tilde{d}_j, \tilde{d}_k\} = 0 &\Rightarrow \left\{ \sum_{j=1}^n (K_{jm}d_m + L_{jm}d_m^*), \sum_{l=1}^n (K_{kl}d_l + L_{kl}d_l^*) \right\} = 0 \quad (3.52) \\ &\Rightarrow \sum_{m,l=1}^n (K_{jm}L_{kl}\{d_m, d_l^*\}) + \sum_{m,l=1}^n (L_{jm}K_{kl}\{d_m^*, d_l\}) = 0 \\ &\Rightarrow \sum_{m=1}^n K_{jm}\overline{L_{km}} + L_{jm}\overline{K_{km}} = 0 \\ &\Rightarrow \sum_{z=1}^{2n} \hat{W}_{jz}\overline{\hat{W}_{(n+k)z}} = 0, \end{aligned}$$

and this shows that \hat{W} is a unitary matrix, which means that $\hat{W}\hat{W}^* = \mathbb{1}$. Thus

$$\begin{bmatrix} K & L \\ \overline{L} & \overline{K} \end{bmatrix} \begin{bmatrix} K^* & L^t \\ L^* & K^t \end{bmatrix} = \mathbb{1} \Rightarrow \begin{cases} KK^* + LL^* = \mathbb{1} \text{ and } \overline{L}L^t + \overline{K}K^t = \mathbb{1} \\ KL^t + LK^t = 0 \text{ and } \overline{L}K^* + \overline{K}L^* = 0. \end{cases} \quad (3.53)$$

Then,

$$\begin{aligned}
\hat{W}J\hat{W}^t &= \begin{bmatrix} K & L \\ \bar{L} & \bar{K} \end{bmatrix} \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} \begin{bmatrix} K^t & L^* \\ L^t & K^* \end{bmatrix} \\
&= \begin{bmatrix} LK^t + KL^t & LL^* + KK^* \\ \bar{K}K^t + \bar{L}L^t & \bar{K}L^* + \bar{L}K^* \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix} = J.
\end{aligned} \tag{3.54}$$

□

In the situation above we say that $\tilde{\mathcal{D}}$ is a Bogoliubov transformation of \mathcal{D} (with respect to the Bogoliubov matrix W). The following Lemma gives a clear idea about the class of Fermionic systems we will deal with.

LEMMA 3.5. *The Bogoliubov transformation between two Fermionic systems is an equivalence relation.*

PROOF. The reflexive property follows from the fact that the identity matrix is the trivial Bogoliubov matrix. The symmetric property is satisfied because the adjoint of a Bogoliubov matrix W is Bogoliubov:

$$WJW^t = J \Rightarrow JW^t = W^*J \Rightarrow J = W^*J\bar{W} \Rightarrow W^*J(W^*)^t = J.$$

The transitive property follows from the fact that Bogoliubov transformation is closed under multiplication: Let W_1 and W_2 be two Bogoliubov matrices then

$$W_1W_2J(W_1W_2)^t = W_1(W_2JW_2^t)W_1^t = W_1JW_1^t = J. \tag{3.55}$$

□

3.2. Quasi-free states

This section is devoted for quasi-free states where we include a definition in terms of Wick's rule, and show that Wick's rule is invariant under Bogoliubov transformation.

We say that the state ρ is *quasi-free* with respect to a Fermionic system \mathcal{D} if for every positive integer m and functions $f_j, g_j : \Lambda \rightarrow \mathbb{C}$ for $j = 1, 2, \dots, m$, we have

$$\left\langle \prod_{j=1}^m D_j \right\rangle_{\rho} = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ \sum_{k=2}^m (-1)^k \langle D_k D_1 \rangle_{\rho} \left\langle \prod_{j=2, j \neq k}^m D_j \right\rangle_{\rho}, & \text{if } m \text{ is even.} \end{cases} \quad (3.56)$$

Here D_j is short for $D(f_j, g_j)$ and a pair $f_j, g_j : \Lambda \rightarrow \mathbb{C}$ as in (3.20). Recall that here and throughout this thesis operator products as on the left hand side of (3.56) are to be read from right to left, i.e. $\prod_{j=1}^m D_j = D_m \dots D_1$. For the case of even m , where (3.56) is a form of *Wick's rule*, an iterative application shows that expectations $\left\langle \prod_{j=1}^m D_j \right\rangle_{\rho}$ can be written as a sum of products of terms of the form $\langle D_r D_s \rangle_{\rho}$. We note that the resulting expression is known as the *Pfaffian* (denoted by pf) of the skew-adjoint $m \times m$ -matrix $D^{(\rho, m)}$ with entries $[D^{(\rho, m)}]_{s,r} = \langle D_r D_s \rangle_{\rho}$ for $1 \leq s < r \leq m$ and appropriately extended by antisymmetry, see Appendix A for a brief introduction to Pfaffians. The Pfaffian of an odd-dimensional skew-adjoint matrix is generally set to zero, so that (3.56) can be restated as

$$\left\langle \prod_{j=1}^m D_j \right\rangle_{\rho} = \text{pf} [D^{(\rho, m)}]. \quad (3.57)$$

The following Lemma states that Wick's rule is invariant under the Bogoliubov transformation and that it is carried over by unitary transformation.

LEMMA 3.6. *Let \mathcal{D} and $\tilde{\mathcal{D}}$ be Fermionic systems on \mathcal{H} and assume that ρ is quasi-free with respect to \mathcal{D} .*

(a) *If $\tilde{\mathcal{D}}$ is a Bogoliubov transformation of \mathcal{D} , then ρ is quasi-free with respect to $\tilde{\mathcal{D}}$.*

(b) *Let U be the unitary relating \mathcal{D} and $\tilde{\mathcal{D}}$ as in Lemma 3.2. Then $U\rho U^*$ is quasi-free with respect to $\tilde{\mathcal{D}}$.*

PROOF. (a) First, let $\{W_{ij}, 1 \leq i, j \leq 2\}$ be the blocks (3.43) of the Bogoliubov matrix \hat{W} as in (3.41), then

$$\begin{aligned}
\tilde{d}(f) &= \sum_{j=1}^n \bar{f}_j \tilde{d}_j \\
&= \sum_{j=1}^n \bar{f}_j \left(\sum_{k=1}^n (W_{11})_{jk} d_k + \sum_{k=1}^n (W_{12})_{jk} d_k^* \right) \\
&= \sum_{k=1}^n \left(\sum_{j=1}^n (W_{11}^t)_{kj} \bar{f}_j \right) d_k + \sum_{k=1}^n \left(\sum_{j=1}^n (W_{12}^t)_{kj} \bar{f}_j \right) d_k^* \\
&= d(W_{11}^* f) + d^*(W_{12}^t \bar{f}).
\end{aligned} \tag{3.58}$$

Similarly,

$$\tilde{d}^*(g) = d(W_{21}^t \bar{g}) + d^*(W_{22}^* g) \tag{3.59}$$

and

$$\tilde{D}(f, g) = \tilde{d}(f) + \tilde{d}^*(g) = d(h) + d^*(r) = D(h, r) \tag{3.60}$$

where

$$h = W_{11}^* f + W_{21}^t \bar{g} \text{ and } r = W_{12}^t \bar{f} + W_{22}^* g. \tag{3.61}$$

Then, if m is even

$$\begin{aligned}
\left\langle \prod_{j=1}^m \tilde{D}(f_j, g_j) \right\rangle_{\rho} &= \left\langle \prod_{j=1}^m D(h_j, r_j) \right\rangle_{\rho} \\
&= \sum_{k=2}^m (-1)^k \langle D(h_k, r_k) D(h_1, r_1) \rangle_{\rho} \left\langle \prod_{j=2, j \neq k}^m D(h_j, r_j) \right\rangle_{\rho} \\
&= \sum_{k=2}^m (-1)^k \langle \tilde{D}(f_k, g_k) \tilde{D}(f_1, g_1) \rangle_{\rho} \left\langle \prod_{j=2, j \neq k}^m \tilde{D}(f_j, g_j) \right\rangle_{\rho}
\end{aligned} \tag{3.62}$$

and if m is odd

$$\left\langle \prod_{j=1}^m \tilde{D}(f_j, g_j) \right\rangle_{\rho} = \left\langle \prod_{j=1}^m D(h_j, r_j) \right\rangle_{\rho} = 0. \tag{3.63}$$

(b) For any positive integer m ,

$$\left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{U\rho U^*} = \left\langle U^* \left(\prod_{j=1}^m \tilde{D}_j \right) U \right\rangle_{\rho} = \left\langle \prod_{j=1}^m D_j \right\rangle_{\rho}. \quad (3.64)$$

Thus, if m is odd, then since ρ is quasi-free with respect to \mathcal{D} then both sides of (3.64) are zeros. If m is even then

$$\begin{aligned} \left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{U\rho U^*} &= \left\langle \prod_{j=1}^m D_j \right\rangle_{\rho} \\ &= \sum_{k=2}^m (-1)^k \langle D_k D_1 \rangle_{\rho} \left\langle \prod_{j=2, j \neq k}^m D_j \right\rangle_{\rho} \\ &= \sum_{k=2}^m (-1)^k \langle \tilde{D}_k \tilde{D}_1 \rangle_{U\rho U^*} \left\langle \prod_{j=2, j \neq k}^m \tilde{D}_j \right\rangle_{U\rho U^*}. \end{aligned} \quad (3.65)$$

□

As an obvious conclusion from Lemma 3.6, if $\tilde{\mathcal{D}}$ and \mathcal{D} belong to the same class of Fermionic systems (under the Bogoliubov transformation relation) and ρ is quasi-free with respect to \mathcal{D} then both $U\rho U^*$ and ρ are quasi-free with respect to both Fermionic systems \mathcal{D} and $\tilde{\mathcal{D}}$. An important class of Fermionic systems is the equivalent class of the Jordan-Wigner Fermionic systems under the Bogoliubov relation. And when a state is quasi-free with respect to this class, we will simply say that the state is quasi-free.

3.2.1. Diagonal states. In this subsection we prove that diagonal states are quasi-free states. First, note that the set of quasi-free states is closed.

LEMMA 3.7. *If $\{\rho_n\}_n$ is a sequence of quasi-free states with respect to the generic Fermionic system $\tilde{\mathcal{D}}$, and $\rho_n \rightarrow \rho$ then ρ is quasi-free with respect to $\tilde{\mathcal{D}}$.*

PROOF. ρ_n is quasi-free means that

$$\left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{\rho_n} = \begin{cases} 0, & \text{if } m \text{ is odd;} \\ \sum_{k=2}^m (-1)^k \langle \tilde{D}_k \tilde{D}_1 \rangle_{\rho_n} \left\langle \prod_{j=2, j \neq k}^m \tilde{D}_j \right\rangle_{\rho_n}, & \text{if } m \text{ is even.} \end{cases} \quad (3.66)$$

Then the statement of the Lemma follows directly by taking the limit as n goes to infinity and using the continuity of the trace. \square

THEOREM 3.2. (Wick's Rule) *The operator $\rho^{(diag)} \in \mathcal{B}(\mathcal{H})$ given by*

$$\rho^{(diag)} = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} \quad (3.67)$$

where, $\eta_j \in \mathbb{R}$ for $1 \leq j \leq n$, satisfies Wicks Rule with respect to the Jordan-Wigner Fermionic system \mathcal{C} .

The strategy of the following proof is essentially the ‘‘classical’’ argument for the proof of Wick’s rule for thermal states in free Fermion systems, e.g. [BR97].

PROOF. We will drop the superscript (*diag*) in the proof for simplicity. For odd values of m in formula (3.56), Wick’s rule follows from proving

$$\left\langle \prod_{j=1}^m c_{r_j}^{\#r_j} \right\rangle_{\rho} = 0. \quad (3.68)$$

We will assume that the product of $c^{\#}$ ’s is not zero, otherwise there is nothing to prove. Since

$$aa^*a = a, \quad a^*aa^* = a^*, \quad \text{and} \quad a^{\#}\sigma^z = \pm a^{\#}$$

then it is easy to see that

$$\prod_{j=1}^m c_{r_j}^{\#r_j} = \pm \bigotimes_{j=1}^n A_j,$$

where $A_j \in \{a_j, a_j^*, a_j a_j^*, a_j^* a_j, \sigma_j^z, \mathbb{1}\}$, and since m is odd then there exists $j_0 \in \{1, 2, \dots, n\}$ such that $A_{j_0} \in \{a_{j_0}, a_{j_0}^*\}$. Then

$$\mathrm{Tr} \left[\prod_{j=1}^m c_{r_j}^{\#r_j} \rho \right] = \pm \prod_{j=1}^n \left(\mathrm{tr} \left[A_j \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} \right] \right).$$

The last product vanishes because

$$\mathrm{tr} \left[a^{\#} \begin{pmatrix} \eta & 0 \\ 0 & 1 - \eta \end{pmatrix} \right] = 0.$$

If m is even then the proof is more involved. First, assume that $\eta_j \notin \{0, 1\}$ for all $1 \leq j \leq n$, and note that

$$c_k \rho = \frac{\eta_k}{1 - \eta_k} \rho c_k, \quad (3.69)$$

because

$$\sigma^z \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} = \begin{pmatrix} \eta_j & 0 \\ 0 & -(1 - \eta_j) \end{pmatrix} = \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix} \sigma^z \quad (3.70)$$

$$a \begin{pmatrix} \eta_k & 0 \\ 0 & 1 - \eta_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \eta_k & 0 \end{pmatrix} = \frac{\eta_k}{1 - \eta_k} \begin{pmatrix} \eta_k & 0 \\ 0 & 1 - \eta_k \end{pmatrix} a \quad (3.71)$$

and since,

$$c(f) \rho = \sum_{k=1}^n \bar{f}_k c_k \rho = \rho \sum_{k=1}^n \bar{f}_k \frac{\eta_k}{1 - \eta_k} c_k. \quad (3.72)$$

Then, we have

$$c(f) \rho = \rho c(D_\eta f), \text{ where } D_\eta := \mathrm{diag} \left\{ \frac{\eta_j}{1 - \eta_j}, j = 1, 2, \dots, n \right\}. \quad (3.73)$$

Similarly, we have

$$c^*(g) \rho = \rho c^*(D_\eta^{-1} g). \quad (3.74)$$

Second, we have

$$\left\{ c(f), C(\tilde{f}, \tilde{g}) \right\} = \langle f, \tilde{g} \rangle_{\ell^2}, \quad \left\{ c^*(g), C(\tilde{f}, \tilde{g}) \right\} = \langle g, \tilde{f} \rangle_{\ell^2}, \quad (3.75)$$

because of the following argument

$$\begin{aligned}
\{c(f), C(\tilde{f}, \tilde{g})\} &= \{c(f), c(\tilde{f})\} + \{c(f), c^*(\tilde{g})\} \\
&= 0 + \left\{ \sum_{k=1}^n \bar{f}_k c_k, \sum_{j=1}^n \tilde{g}_j c_j^* \right\} \\
&= \sum_{j,k=1}^n \bar{f}_k \tilde{g}_j \{c_k, c_j^*\} \\
&= \langle f, \tilde{g} \rangle_{\ell^2}.
\end{aligned}$$

The statement with $c^*(g)$ can be shown using a similar approach. Now, if m is even then

$$\left\langle \prod_{j=1}^m C_j \right\rangle_{\rho} = \left\langle \left(\prod_{j=2}^m C_j \right) c(f_1) \right\rangle_{\rho} + \left\langle \left(\prod_{j=2}^m C_j \right) c^*(g_1) \right\rangle_{\rho}. \quad (3.76)$$

The first term equals

$$\begin{aligned}
&= \text{Tr} \left[\prod_{j=2}^m C_j c(f_1) \rho \right] \stackrel{(3.73)}{=} \text{Tr} \left[\prod_{j=2}^m C_j \rho c(D_{\eta} f_1) \right] \\
&= \left\langle c(D_{\eta} f_1) \left(\prod_{j=2}^m C_j \right) \right\rangle_{\rho} = \left\langle c(D_{\eta} f_1) C_m \left(\prod_{j=2}^{m-1} C_j \right) \right\rangle_{\rho} \\
&= \left\langle \{c(D_{\eta} f_1), C_m\} \left(\prod_{j=2}^{m-1} C_j \right) \right\rangle_{\rho} - \left\langle C_m c(D_{\eta} f_1) \left(\prod_{j=2}^{m-1} C_j \right) \right\rangle_{\rho} \\
&= \stackrel{(3.75)}{\langle D_{\eta} f_1, g_m \rangle_{\ell^2}} \left\langle \prod_{j=2}^{m-1} C_j \right\rangle_{\rho} - \left\langle C_m c(D_{\eta} f_1) \left(\prod_{j=2}^{m-1} C_j \right) \right\rangle_{\rho} \\
&= \langle D_{\eta} f_1, g_m \rangle_{\ell^2} \left\langle \prod_{j=2}^{m-1} C_j \right\rangle_{\rho} - \left\langle C_m c(D_{\eta} f_1) C_{m-1} \left(\prod_{j=2}^{m-2} C_j \right) \right\rangle_{\rho} \\
&= \langle D_{\eta} f_1, g_m \rangle_{\ell^2} \left\langle \prod_{j=2}^{m-1} C_j \right\rangle_{\rho} - \left[\left\langle C_m \{c(D_{\eta} f_1), C_{m-1}\} \left(\prod_{j=2}^{m-2} C_j \right) \right\rangle_{\rho} - \right. \\
&\quad \left. \left\langle C_m C_{m-1} c(D_{\eta} f_1) \left(\prod_{j=2}^{m-2} C_j \right) \right\rangle_{\rho} \right]
\end{aligned}$$

$$= \sum_{k=2}^m (-1)^k \langle D_\eta f_1, g_k \rangle_{\ell^2} \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho - \left\langle \left(\prod_{j=2}^m C_j \right) c(D_\eta f_1) \right\rangle_\rho.$$

Then, by taking the second term to the left hand side and defining $\tilde{f}_1 := (\mathbb{1} + D_\eta)f_1$, we get

$$\left\langle \left(\prod_{j=1}^m C_j \right) c(\tilde{f}_1) \right\rangle_\rho = \sum_{k=2}^m (-1)^k \langle D_\eta (\mathbb{1} + D_\eta)^{-1} \tilde{f}_1, g_k \rangle_{\ell^2} \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho. \quad (3.77)$$

A simple calculation gives that

$$\langle D_\eta (\mathbb{1} + D_\eta)^{-1} \tilde{f}_1, g_k \rangle_{\ell^2} = \sum_{j=1}^n \eta_j \overline{\tilde{f}_{1j}} g_{kj}, \quad (3.78)$$

which match with $\langle C(f_k, g_k) c(\tilde{f}_1) \rangle_\rho$ as follows

$$\begin{aligned} \left\langle C(f_k, g_k) c(\tilde{f}_1) \right\rangle_\rho &= \langle c(f_k) c(\tilde{f}_1) \rangle_\rho + \langle c^*(g_k) c(\tilde{f}_1) \rangle_\rho \\ &= \sum_{j=1}^n \overline{\tilde{f}_{1j}} g_{kj} \langle c_j^* c_j \rangle_\rho = \sum_{j=1}^n \eta_j \overline{\tilde{f}_{1j}} g_{kj}. \end{aligned} \quad (3.79)$$

Thus equation (3.77) becomes

$$\left\langle \left(\prod_{j=2}^m C_j \right) c(\tilde{f}_1) \right\rangle_\rho = \sum_{k=2}^m (-1)^k \langle C(f_k, g_k) c(\tilde{f}_1) \rangle_\rho \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho. \quad (3.80)$$

By applying similar steps to the second term of (3.76) we get

$$\begin{aligned} \left\langle \left(\prod_{j=2}^m C_j \right) c^*(g_1) \right\rangle_\rho &= \left\langle c^*(D_\eta^{-1} g_1) \left(\prod_{j=2}^m C_j \right) \right\rangle_\rho \\ &= \sum_{k=2}^m (-1)^k \langle D_\eta^{-1} g_1, f_k \rangle_{\ell^2} \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho - \left\langle \left(\prod_{j=2}^m C_j \right) c^*(D_\eta^{-1} g_1) \right\rangle_\rho. \end{aligned}$$

Then by moving the second term to the left hand side and defining $\tilde{g}_1 := (\mathbb{1} + D_\eta^{-1}) g_1$, and noting that

$$\langle C(f_k, g_k)^* c(g_1) \rangle_\rho = \langle c(f_k) c^*(g_1) \rangle_\rho + \langle c^*(g_k) c^*(g_1) \rangle_\rho \quad (3.81)$$

$$\begin{aligned}
&= \sum_j \overline{f_{k_j}} g_{1_j} \langle c_j c_j^* \rangle_\rho = \sum_j (1 - \eta_j) \overline{f_{k_j}} g_{k_j} \\
&= \langle D_\eta^{-1} (\mathbb{1} + D_\eta^{-1})^{-1} \tilde{g}_1, f_k \rangle_{\ell^2}.
\end{aligned}$$

Then we get

$$\left\langle \left(\prod_{j=2}^m C_j \right) c^*(\tilde{g}_1) \right\rangle_\rho = \sum_{k=2}^m (-1)^k \langle C(f_k, g_k) c^*(\tilde{g}_1) \rangle_\rho \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho. \quad (3.82)$$

By substituting the results (3.80) and (3.82) in (3.76), we get our result

$$\left\langle \prod_{j=1}^m C_j \right\rangle_\rho = \sum_{j=2}^m (-1)^k \langle C_k C_1 \rangle_\rho \left\langle \prod_{j=2, j \neq k}^m C_j \right\rangle_\rho. \quad (3.83)$$

Finally, in the general case where $\eta_j \in \mathbb{R}$ for all $j = 1, 2, \dots, n$, there exists a sequence

$$\rho_n = \bigotimes_{j=1}^n \begin{pmatrix} \eta^{(n)}_j & 0 \\ 0 & 1 - \eta^{(n)}_j \end{pmatrix} \quad (3.84)$$

where $\eta^{(n)}_j \notin \{0, 1\} \rightarrow \eta_j$ as $n \rightarrow \infty$ and thus $\rho_n \rightarrow \rho$. Now since ρ_n is quasi-free with respect to the \mathcal{C} then ρ is quasi-free with respect to the Fermionic system \mathcal{C} by Lemma 3.7. \square

Let \mathcal{B} be any Fermionic system in $\mathcal{B}(\mathcal{H})$. Corollary 3.2 below shows that the density matrices corresponding to the ONBs of \mathcal{H}

$$\psi_\alpha = (b_1^*)^{\alpha_1} (b_2^*)^{\alpha_2} \dots (b_n^*)^{\alpha_n} \Omega, \quad \alpha \in \{0, 1\}^n,$$

are quasi-free with respect to the Fermionic system \mathcal{B} . Here Ω is the vacuum vector of the operators b_j .

COROLLARY 3.2. *The density matrices $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ are quasi-free with respect to the Fermionic system \mathcal{B} .*

PROOF. Let Ω_c and $\psi_\alpha^{(c)} = (c_1^*)^{\alpha_1} \dots (c_n^*)^{\alpha_n} \Omega_c$ be the vacuum and eigenstates associated with \mathcal{C} . By Lemma 3.6(b) it is enough to prove that $\rho_\alpha^{(c)} := |\psi_\alpha^{(c)}\rangle\langle\psi_\alpha^{(c)}|$ is

quasi-free with respect to \mathcal{C} for all α . This can be done by finding $\rho_\alpha^{(c)}$ explicitly and find that they are diagonal density matrices. From the definition (3.34) of the c_j we see that $c_j(e_\downarrow)^{\otimes n} = 0$ for all j . This gives that, up to a phase, $\Omega_c = (e_\downarrow)^{\otimes n}$ is the vacuum for \mathcal{C} , and therefore

$$\rho_0^{(c)} = |\Omega_c\rangle\langle\Omega_c| = \bigotimes_{j=1}^n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \prod_{j=1}^n c_j c_j^*. \quad (3.85)$$

Since for $1 \leq k \leq n$ we have

$$c_k^* \left(\prod_{j=1}^n c_j c_j^* \right) c_k = \left(\prod_{j=1, j \neq k}^n c_j c_j^* \right) c_k^* c_k c_k^* c_k = \left(\prod_{j=1, j \neq k}^n c_j c_j^* \right) c_k^* c_k, \quad (3.86)$$

it is easy to see that

$$\begin{aligned} \rho_\alpha^{(c)} &= \prod_{j=1}^n (c_{n-j}^*)^{\alpha_{n-j}} |\Omega_c\rangle\langle\Omega_c| \prod_{j=1}^n (c_j)^{\alpha_j} = \prod_{j=1, \alpha_j=0}^n c_j c_j^* \prod_{j=1, \alpha_j=1}^n c_j^* c_j \quad (3.87) \\ &= \bigotimes_{j=1}^n \begin{pmatrix} \delta_{\alpha_j,1} & 0 \\ 0 & \delta_{\alpha_j,0} \end{pmatrix}, \end{aligned}$$

where δ is the Kronecker delta function. Thus, as a product of diagonal states, $\rho_\alpha^{(c)}$ is quasi-free with respect to \mathcal{C} by Proposition 3.2. \square

3.2.2. Reduced states of quasi-free states. The goal of this subsection is to prove Theorem 3.3. We consider the Hilbert space $\mathcal{H} = \bigotimes_{j \in \Lambda} \mathbb{C}^2$ where $\Lambda = [1, n] := \{1, 2, \dots, n\}$. We distinguish a subinterval $\Lambda_0 = [r, r + \ell - 1] \subset \Lambda$ and decompose the Hilbert space as $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where

$$\mathcal{H}_1 = \bigotimes_{j \in \Lambda_0} \mathbb{C}^2, \text{ and } \mathcal{H}_2 = \bigotimes_{j \in \Lambda \setminus \Lambda_0} \mathbb{C}^2. \quad (3.88)$$

Let ρ be a quasi-free state with respect to \mathcal{C} . We will show that the reduced state $\rho^1 = \text{Tr}_{\mathcal{H}_2} \rho$ is quasi-free with respect to the local Jordan-Wigner Fermionic system \mathcal{C}_1 , see Section 3.1.1 for the definition. First, we consider the Jordan-Wigner operators

defined on Λ_0 as in (3.36), then define their extension to \mathcal{H} as follows

$$\tilde{c}_j := \mathbb{1}^{\otimes(r-1)} \otimes c_j^{(1)} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)}. \quad (3.89)$$

In what follows we list the relations between these extended states, the local Jordan-Wigner operators, and the Jordan-Wigner operators defined over the whole chain Λ . For $f : \Lambda_0 \rightarrow \mathbb{C}$, we use the following notations,

$$c^{(1)}(f) = \sum_{j \in \Lambda_0} \bar{f}_j c_j^{(1)}, \quad \tilde{c}_r(f) = \sum_{j \in \Lambda_0} \bar{f}_j \tilde{c}_j, \quad c_r(f) = \sum_{j \in \Lambda_0} \bar{f}_j c_j. \quad (3.90)$$

Similarly we define $c^{(1)*}(g)$, $\tilde{c}_r^*(g)$, and $c_r^*(g)$ for $g : \Lambda_0 \rightarrow \mathbb{C}$. We also use the following notations: for $f, g : \Lambda_0 \rightarrow \mathbb{C}$

$$C^{(1)}(f, g) = c^{(1)}(f) + c^{(1)}(g), \quad \tilde{C}_r(f, g) = \tilde{c}_r(f) + \tilde{c}_r(g), \quad C_r(f, g) = c_r(f) + c_r^*(g). \quad (3.91)$$

Note that $C_r(f_j, g_j) = C(\tilde{f}_j, \tilde{g}_j)$ for some $\tilde{f}_j, \tilde{g}_j : \Lambda \rightarrow \mathbb{C}$ that are extensions by zeros of f_j and g_j ; respectively. We include some relations between these operators in the following Lemma

LEMMA 3.8. *For $j \in \Lambda_0$, and $f, g, f_1, g_1, f_2, g_2 : \Lambda_0 \rightarrow \mathbb{C}$ we have*

- (a) $\tilde{C}_r(f_j, g_j) = (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) C_r(f_j, g_j)$.
- (b) $\tilde{c}_j^\# \tilde{c}_k^\# = c_j^\# c_k^\#$.
- (c) $\tilde{c}_r^\#(f) \tilde{c}_r^\#(g) = c_r^\#(f) c_r^\#(g)$, and hence

$$\tilde{C}_r(f_1, g_1) \tilde{C}_r(f_2, g_2) = C_r(f_1, g_1) C_r(f_2, g_2).$$

PROOF. (a) follows from

$$\begin{aligned} c_j &= (\sigma^z)^{\otimes(r-1)} \otimes c_j^{(1)} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)} \\ &= (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) \left(\mathbb{1}^{\otimes(r-1)} \otimes c_j^{(1)} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)} \right) \\ &= (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) \tilde{c}_j. \end{aligned} \quad (3.92)$$

Then multiplying both side by $(\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z)$ will give that

$$\tilde{c}_j = (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) c_j. \quad (3.93)$$

Then

$$\begin{aligned} \tilde{C}_r(f_j, g_j) &= \sum_{j \in \Lambda_0} \bar{f}_j \tilde{c}_j + \sum_{j \in \Lambda_0} g_j \tilde{c}_j^* \\ &= (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) C_r(f_j, g_j). \end{aligned}$$

(b) Note the following

$$\begin{aligned} \tilde{c}_j^\# \tilde{c}_k^\# &= \left(\mathbb{1}^{\otimes(r-1)} \otimes c_j^{(1)\#} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)} \right) \left(\mathbb{1}^{\otimes(r-1)} \otimes c_k^{(1)\#} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)} \right) \\ &= \mathbb{1}^{\otimes(r-1)} \otimes c_j^{(1)\#} c_k^{(1)\#} \otimes \mathbb{1}^{\otimes(N-\ell-r+1)} \\ &= c_j^\# c_k^\#. \end{aligned}$$

(c) We will do one case and the others are identical,

$$\begin{aligned} \tilde{c}_r(f) \tilde{c}_r^*(g) &= \left(\sum_{j \in \Lambda_0} \bar{f}_j \tilde{c}_j \right) \left(\sum_{j \in \Lambda_0} g_j \tilde{c}_j^* \right) = \sum_{j, k \in \Lambda_0} \bar{f}_j g_k \tilde{c}_j \tilde{c}_k^* \quad (3.94) \\ &= \sum_{j, k \in \Lambda_0} \bar{f}_j g_k c_j c_k^* = \left(\sum_{j \in \Lambda_0} \bar{f}_j c_j \right) \left(\sum_{j \in \Lambda_0} g_j c_j^* \right) \\ &= c_r(f) c_r^*(g). \end{aligned}$$

The second statement in (c) is a generalization of the first:

$$\begin{aligned} \tilde{C}_r(f_1, g_1) \tilde{C}_r(f_2, g_2) &= (\tilde{c}_r(f_1) + \tilde{c}_r^*(g_1)) (\tilde{c}_r(f_2) + \tilde{c}_r^*(g_2)) \\ &= \tilde{c}_r(f_1) \tilde{c}_r(f_2) + \tilde{c}_r(f_1) \tilde{c}_r^*(g_2) + \tilde{c}_r^*(g_1) \tilde{c}_r(f_2) + \tilde{c}_r^*(g_1) \tilde{c}_r^*(g_2) \\ &= c_r(f_1) c_r(f_2) + c_r(f_1) c_r^*(g_2) + c_r^*(g_1) c_r(f_2) + c_r^*(g_1) c_r^*(g_2) \\ &= C_r(f_1, g_1) C_r(f_2, g_2). \end{aligned}$$

□

THEOREM 3.3. *If ρ is quasi-free then the reduced state $\rho^1 := \text{Tr}_{\mathcal{H}_2} \rho$ is quasi-free with respect to the local Jordan-Wigner system \mathcal{C}_1 (Associated with \mathcal{H}_1).*

PROOF. This is equivalent to proving that ρ is quasi-free with respect to the \tilde{c}_j for $j \in \Lambda_0$, because of the following: For any positive integer m and functions $f_j, g_j : \Lambda_0 \rightarrow \mathbb{C}$ for $j \in \Lambda_0$ we have,

$$\begin{aligned} \left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho^1} &= \text{Tr} \left[\prod_{j=1}^m C^{(1)}(f_j, g_j) \text{Tr}_{\mathcal{H}_2} \rho \right] \\ &= \text{Tr} \left[\prod_{j=1}^m \tilde{C}_r(f_j, g_j) \rho \right] \\ &= \left\langle \prod_{j=1}^m \tilde{C}_r(f_j, g_j) \right\rangle_{\rho}. \end{aligned}$$

Now, if m is even then using Lemma 3.8, we get that

$$\prod_{j=1}^m \tilde{C}_r(f_j, g_j) = \prod_{j=1}^m C_r(f_j, g_j). \quad (3.95)$$

Thus,

$$\left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho^1} = \left\langle \prod_{j=1}^m C_r(f_j, g_j) \right\rangle_{\rho} = \text{pf} [C_r^{(\rho, m)}], \quad (3.96)$$

where $C_r^{(\rho, m)}$ is the $m \times m$ anti-symmetric matrix with entries

$$[C_r^{(\rho, m)}]_{j,k} = \langle C_r(f_k, g_k) C_r(f_j, g_j) \rangle_{\rho}, \text{ for } 1 \leq j < k \leq m. \quad (3.97)$$

Since

$$\begin{aligned} \langle C_r(f_k, g_k) C_r(f_j, g_j) \rangle_{\rho} &= \langle \tilde{C}_r(f_k, g_k) \tilde{C}_r(f_j, g_j) \rangle_{\rho} \\ &= \langle C^{(1)}(f_k, g_k) C^{(1)}(f_j, g_j) \rangle_{\rho^1} := [C^{(1)(\rho^1, m)}]_{j,k}. \end{aligned}$$

Thus, Wick's rule is satisfied if m is even:

$$\left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho^1} = \text{pf} [C^{(1)(\rho^1, m)}].$$

If m is odd then using (3.95), Lemma 3.8, and partial trace property in Lemma 2.3(c) we get,

$$\begin{aligned}
\left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho^1} &= \left\langle \tilde{C}_r(f_m, g_m) \prod_{j=1}^{m-1} \tilde{C}_r(f_j, g_j) \right\rangle_{\rho} \\
&= \left\langle \tilde{C}_r(f_m, g_m) \prod_{j=1}^{m-1} C_r(f_j, g_j) \right\rangle_{\rho} \\
&= \left\langle (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) C_r(f_m, g_m) \prod_{j=1}^{m-1} C_r(f_j, g_j) \right\rangle_{\rho} \\
&= \left\langle (\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z) \prod_{j=1}^m C_r(f_j, g_j) \right\rangle_{\rho}.
\end{aligned}$$

Now note the following

$$\begin{aligned}
\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z &= \prod_{j=1}^{r-1} \sigma_j^z = \prod_{j=1}^{r-1} (a_j^* a_j - a_j a_j^*) \\
&= \prod_{j=1}^{r-1} (c_j^* c_j - c_j c_j^*) = \prod_{j=1}^{r-1} (c_j + c_j^*)(c_j - c_j^*).
\end{aligned} \tag{3.98}$$

Thus, $\sigma_1^z \sigma_2^z \dots \sigma_{r-1}^z$ is an even product of $C(\tilde{f}_j, \tilde{g}_j)$'s with suitable choice of \tilde{f}_j and \tilde{g}_j , and since ρ is quasi-free with respect to \mathcal{C} then if m is odd we have

$$\left\langle \prod_{j=1}^m C^{(1)}(f_j, g_j) \right\rangle_{\rho^1} = 0. \tag{3.99}$$

□

3.2.3. Tensor product of quasi-free states. In this subsection we consider a partition of Λ into $\{\Lambda^{(k)}, k = 1, 2, \dots, m\}$, associated with the Hilbert spaces $\{\mathcal{H}^{(k)}\}$. By $\mathcal{C}^{(k)}$ we denote the local Jordan-Wigner Fermionic system corresponding to $\Lambda^{(k)}$. Let $\{\rho_k \in \mathcal{B}(\mathcal{H}^{(k)}), k = 1, 2, \dots, m\}$ be an arbitrary set of density matrices.

LEMMA 3.9. *If ρ_k is quasi-free with respect to \mathcal{C}_k for $k = 1, 2, \dots, m$. Then $\bigotimes_{k=1}^m \rho_k$ is quasi-free.*

PROOF. It is enough to prove this for $m = 2$ then Lemma 3.9 will follow by an iterative process. It is easy to see that

$$c_j = \begin{cases} c_j^{(1)} \otimes \mathbb{1} & \text{if } j \in \Lambda^{(1)} \\ (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes c_j^{(2)}, & \text{if } j \in \Lambda^{(2)} \end{cases} \quad (3.100)$$

A direct calculation gives

$$c_j^\# c_k^\# = \begin{cases} (c_j^{(1)})^\# (c_k^{(1)})^\# \otimes \mathbb{1}, & \text{if } j, k \in \Lambda^{(1)}; \\ \mathbb{1} \otimes (c_j^{(2)})^\# (c_k^{(2)})^\#, & \text{if } j, k \in \Lambda^{(2)}; \\ (c_j^{(1)})^\# (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes (c_k^{(2)})^\#, & \text{if } j \in \Lambda^{(1)}, k \in \Lambda^{(2)}; \\ (\sigma^z)^{\otimes |\Lambda^{(1)}|} (c_k^{(1)})^\# \otimes (c_j^{(2)})^\# & \text{if } k \in \Lambda^{(1)}, j \in \Lambda^{(2)}. \end{cases} \quad (3.101)$$

Our goal is to prove that for any positive integer m and functions $f_j, g_j : \Lambda \rightarrow \mathbb{C}$ for $j = 1, 2, \dots, m$, we have

$$\left\langle \prod_{j=1}^m C(f_j, g_j) \right\rangle_{\rho_1 \otimes \rho_2} = \text{pf} [C^{(\rho_1 \otimes \rho_2, m)}] \quad (3.102)$$

where

$$C(f_j, g_j) = \sum_{k \in \Lambda} \overline{f_{j,k}} c_k + \sum_{k \in \Lambda} g_{j,k} c_k^* \quad (3.103)$$

and $C^{(\rho_1 \otimes \rho_2, m)}$ is the $m \times m$ anti-symmetric matrix whose elements are

$$[C^{(\rho_1 \otimes \rho_2, m)}]_{j,k} = \langle C(f_k, g_k) C(f_j, g_j) \rangle_{\rho_1 \otimes \rho_2}, \text{ for } 1 \leq j < k \leq m \quad (3.104)$$

and extended appropriately by antisymmetry. First, we need to introduce the following:

For functions $f_1, g_1 : \Lambda^{(1)} \rightarrow \mathbb{C}$ and $f_2, g_2 : \Lambda^{(2)} \rightarrow \mathbb{C}$, define

$$C_{(1)}(f_1, g_1) := \sum_{j \in \Lambda^{(1)}} \overline{f_{1,j}} c_j^{(1)} + \sum_{k \in \Lambda^{(1)}} g_{1,k} (c_k^{(1)})^* \quad (3.105)$$

$$C_{(2)}(f_2, g_2) := \sum_{j \in \Lambda^{(2)}} \overline{f_{2,j}} c_j^{(2)} + \sum_{k \in \Lambda^{(2)}} g_{2,k} (c_k^{(2)})^*. \quad (3.106)$$

Using (3.100), for $f, g : \Lambda \rightarrow \mathbb{C}$, we get

$$\begin{aligned}
C(f, g) &= \left(\sum_{k \in \Lambda^{(1)}} \overline{f_k} c_k + \sum_{k \in \Lambda^{(1)}} g_k c_k^* \right) + \left(\sum_{k \in \Lambda^{(2)}} \overline{f_k} c_k + \sum_{k \in \Lambda^{(2)}} g_k c_k^* \right) \quad (3.107) \\
&= \left(\sum_{k \in \Lambda^{(1)}} \overline{f_k} c_k^{(1)} + \sum_{k \in \Lambda^{(1)}} g_k (c_k^{(1)})^* \right) \otimes \mathbb{1} + \\
&\quad (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes \left(\sum_{k \in \Lambda^{(2)}} \overline{f_k} c_k^{(2)} + \sum_{k \in \Lambda^{(2)}} g_k (c_k^{(2)})^* \right) \\
&= C_{(1)}(f^{(1)}, g^{(1)}) \otimes \mathbb{1} + (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes C_{(2)}(f^{(2)}, g^{(2)})
\end{aligned}$$

where $f^{(1)} := f|_{\Lambda^{(1)}}$, $f^{(2)} := f|_{\Lambda^{(2)}}$. For simplicity, we will make the following definitions:

$$A := C_{(1)}(f^{(1)}, g^{(1)}) \otimes \mathbb{1} \quad (3.108)$$

$$B := (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes C_{(2)}(f^{(2)}, g^{(2)}). \quad (3.109)$$

Note that

$$\{A, B\} = 0, \quad (3.110)$$

this follows directly from $\{(c_j^{(1)})^\#, (\sigma^z)^{\otimes |\Lambda^{(1)}|}\} = 0$. We are now ready to prove Wick's rule: For any positive integer m and functions $f_j, g_j : \Lambda \rightarrow \mathbb{C}$, for $j = 1, 2, \dots, m$,

$$\begin{aligned}
\left\langle \prod_{j=1}^m C(f_j, g_j) \right\rangle_{\rho_1 \otimes \rho_2} &= \left\langle \prod_{j=1}^m (A_j + B_j) \right\rangle_{\rho_1 \otimes \rho_2} \quad (3.111) \\
&= \sum_{\alpha \in \{0,1\}^m} \left\langle \prod_{j=1}^m (A_j^{\alpha_j} B_j^{1-\alpha_j}) \right\rangle_{\rho_1 \otimes \rho_2}.
\end{aligned}$$

Now, let's try to write

$$\left\langle \prod_{j=1}^m (A_j^{\alpha_j} B_j^{1-\alpha_j}) \right\rangle_{\rho_1 \otimes \rho_2} \quad (3.112)$$

in an explicit form. Let $J := \{j, \alpha_j = 1\}$, then

(a) If $|J|$ is odd, i.e. the A_j 's appear an odd number of times, then (3.112) vanishes

because the following two facts

$$\left\langle \prod_{j \in J} C_{(1)}(f_j^{(1)}, g_j^{(1)}) \right\rangle_{\rho_1} = 0 \quad (3.113)$$

$$\left\langle \prod_{j=1}^m \left(C_{(1)}(f_j^{(1)}, g_j^{(1)}) \right)^{\alpha_j} \left((\sigma^z)^{\otimes |\Lambda^{(1)}|} \right)^{1-\alpha_j} \right\rangle_{\rho_1} = 0. \quad (3.114)$$

The first equality holds because ρ_1 is quasi-free with respect to $\mathcal{C}^{(1)}$. In the second equality, $\sigma_j^z = (c_j^{(1)} + (c_j^{(1)})^*)(c_j^{(1)} - (c_j^{(1)})^*)$ for $j \in \Lambda^{(1)}$, thus we still have a multiplication of an odd number of $C_{(1)}(\tilde{f}_j^{(1)}, \tilde{g}_j^{(1)})$'s, for some functions $\tilde{f}_j, \tilde{g}_j : \Lambda \rightarrow \mathbb{C}$.

(b) If $|J^c|$ is odd, i.e. the B_j 's appear an odd number of times, then (3.112) vanishes because

$$\left\langle \prod_{j \in J^c} C_{(2)}(f_j^{(2)}, g_j^{(2)}) \right\rangle_{\rho_2} = 0. \quad (3.115)$$

Thus, the only case where (3.112) does not vanish is when m and $|J|$ are even. For this case we will prove that

$$\prod_{j=1}^m \left(A_j^{\alpha_j} B_j^{1-\alpha_j} \right) = (-1)^{\sum_{j \in J} j + \frac{|J|}{2}} \prod_{j \in J} A_j \prod_{k \in J^c} B_k. \quad (3.116)$$

The products in the right hand side of (3.116) should be understood as follows:

$$\prod_{j \in J} A_j := A_{j_{|J|}} A_{j_{(|J|-1)}} \dots A_{j_2} A_{j_1}, \text{ where } J = \{j_k, k = 1, 2, \dots, |J|\},$$

and $j_1 < j_2 < \dots < j_{|J|}$. The product in the left hand side of (3.116) has the following general form

$$\dots A_{j_{|J|}} \dots A_{j_{(|J|-1)}} \dots A_{j_2} \dots A_{j_1} \dots,$$

with possible B 's in between the A 's and on the two ends. We will pull the A 's to the right, one after the other starting from A_{j_1} until the last one $A_{j_{|J|}}$, each step will create a negative. Note that A_{j_1} will need to move $j_1 - 1$ steps, A_{j_2} will need to move $j_2 - 2$ steps, and in general A_{j_k} will move $j_k - k$ steps until we order them as in the right hand side of (3.116) (actually the opposite way but note that the two big

products commute). Thus the total number of steps is

$$\sum_{k=1}^{|J|} (j_k - k) = \sum_{j \in J} j - \frac{|J|(|J| + 1)}{2}$$

but $(|J| + 1)$ is odd, thus

$$(-1)^{\sum_{j \in J} j - \frac{|J|(|J| + 1)}{2}} = (-1)^{\sum_{j \in J} j} \left((-1)^{(|J| + 1)} \right)^{-\frac{|J|}{2}} = (-1)^{\sum_{j \in J} j - \frac{|J|}{2}} = (-1)^{\sum_{j \in J} j + \frac{|J|}{2}}.$$

Then we proceed as follows:

$$\begin{aligned} & \left\langle \prod_{j=1}^m \left(A_j^{\alpha_j} B_j^{1-\alpha_j} \right) \right\rangle_{\rho_1 \otimes \rho_2} & (3.117) \\ & = (-1)^{\sum_{j \in J} j + \frac{|J|}{2}} \left\langle \prod_{j \in J} C_{(1)}(f_j^{(1)}, g_j^{(1)}) \right\rangle_{\rho_1} \left\langle \prod_{k \in J^c} C_{(2)}(f_k^{(2)}, g_k^{(2)}) \right\rangle_{\rho_2} \\ & = (-1)^{\sum_{j \in J} j + \frac{|J|}{2}} \text{pf} \left[C_{(1)}^{(\rho_1, J)} \right] \text{pf} \left[C_{(2)}^{(\rho_2, J^c)} \right] \end{aligned}$$

where $C_{(1)}^{(\rho_1, J)}$ is an anti-symmetric $|J| \times |J|$ matrix, whose jk -th element (for $j, k \in J$ and $j < k$) is

$$[C_{(1)}^{(\rho_1, J)}]_{jk} = \langle C_{(1)}(f_k^{(1)}, g_k^{(1)}) C_{(1)}(f_j^{(1)}, g_j^{(1)}) \rangle_{\rho_1} \quad (3.118)$$

and extended appropriately by antisymmetry. Similarly, $C_{(2)}^{(\rho_2, J^c)}$ is an anti-symmetric $|J^c| \times |J^c|$ matrix, whose jk -th element (for $j, k \in J^c$ and $j < k$) is

$$[C_{(2)}^{(\rho_2, J^c)}]_{jk} = \langle C_{(2)}(f_k^{(2)}, g_k^{(2)}) C_{(2)}(f_j^{(2)}, g_j^{(2)}) \rangle_{\rho_2}, \quad (3.119)$$

and extended appropriately by antisymmetry. Note that $C_{(1)}^{(\rho_1, J)}$ is a restriction of $C_{(1)}^{(\rho_1, m)}$ to $\text{span}\{e_j, j \in J\}$. Similarly, $C_{(2)}^{(\rho_2, J^c)}$ is a restriction of $C_{(2)}^{(\rho_2, m)}$ to $\text{span}\{e_j, j \in J^c\}$, where $\text{pf}[\emptyset] := 1$. Thus,

$$\sum_{\alpha \in \{0,1\}^m} \left\langle \prod_{j=1}^m \left(A_j^{\alpha_j} B_j^{1-\alpha_j} \right) \right\rangle_{\rho_1 \otimes \rho_2} \quad (3.120)$$

$$\begin{aligned}
&= \sum_{\substack{J \subset \{1, 2, \dots, m\} \\ |J|: \text{ even}}} (-1)^{\sum_{j \in J} j + \frac{|J|}{2}} \text{pf} \left[C_{(1)}^{(\rho_1, J)} \right] \text{pf} \left[C_{(2)}^{(\rho_2, J^c)} \right] \\
&= \text{pf} \left[C_{(1)}^{(\rho_1, m)} + C_{(2)}^{(\rho_2, m)} \right]
\end{aligned}$$

where we used in the last step the following identity about pfaffians:

$$\text{pf}[A + B] = \sum_{\substack{J \subset \{1, 2, \dots, n\} \\ |J|: \text{ even}}} (-1)^{\sum_{j \in J} j + \frac{|J|}{2}} \text{pf}[A_J] \text{pf}[B_{J^c}] \quad (3.121)$$

for $n \times n$ anti-symmetric matrices A and B , A_J means the restriction of A to $\text{span}\{e_j, j \in J\}$, see for example Lemma 4.2 in [Ste90] or [MM60]. In the following, we will prove that

$$C_{(1)}^{(\rho_1, m)} + C_{(2)}^{(\rho_2, m)} = C^{(\rho_1 \otimes \rho_2, m)}. \quad (3.122)$$

Using (3.107), we get

$$\begin{aligned}
\langle C(f_k, g_k)C(f_j, g_j) \rangle_{\rho_1 \otimes \rho_2} &= \langle (A_k + B_k)(A_j + B_j) \rangle_{\rho_1 \otimes \rho_2} \\
&= \langle A_k A_j \rangle_{\rho_1 \otimes \rho_2} + \langle A_k B_j \rangle_{\rho_1 \otimes \rho_2} + \langle B_k A_j \rangle_{\rho_1 \otimes \rho_2} + \langle B_k B_j \rangle_{\rho_1 \otimes \rho_2},
\end{aligned}$$

and note that

$$\begin{aligned}
\langle A_k B_j \rangle_{\rho_1 \otimes \rho_2} &= \left\langle C_{(1)}(f_k^{(1)}, g_k^{(1)})(\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes C_{(2)}(f_j^{(2)}, g_j^{(2)}) \right\rangle_{\rho_1 \otimes \rho_2} \\
&= \left\langle C_{(1)}(f_k^{(1)}, g_k^{(1)})(\sigma^z)^{\otimes |\Lambda^{(1)}|} \right\rangle_{\rho_1} \left\langle C_{(2)}(f_j^{(2)}, g_j^{(2)}) \right\rangle_{\rho_2} \\
&= 0.
\end{aligned}$$

A similar argument gives that

$$\begin{aligned}
\langle B_k A_j \rangle_{\rho_1 \otimes \rho_2} &= 0 \\
\langle A_k A_j \rangle_{\rho_1 \otimes \rho_2} &= \left\langle C_{(1)}(f_k^{(1)}, g_k^{(1)})C_{(1)}(f_j^{(1)}, g_j^{(1)}) \right\rangle_{\rho_1}
\end{aligned}$$

$$\langle B_k B_j \rangle_{\rho_1 \otimes \rho_2} = \left\langle C_{(2)}(f_k^{(2)}, g_k^{(2)}) C_{(2)}(f_j^{(2)}, g_j^{(2)}) \right\rangle_{\rho_2}.$$

Thus, for $1 \leq j < k \leq m$, we get

$$\begin{aligned} [C^{(\rho_1 \otimes \rho_2, m)}]_{jk} &= \langle C(f_k, g_k) C(f_j, g_j) \rangle_{\rho_1 \otimes \rho_2} \\ &= \left\langle C_{(1)}(f_k^{(1)}, g_k^{(1)}) C_{(1)}(f_j^{(1)}, g_j^{(1)}) \right\rangle_{\rho_1} + \left\langle C_{(2)}(f_k^{(2)}, g_k^{(2)}) C_{(2)}(f_j^{(2)}, g_j^{(2)}) \right\rangle_{\rho_2} \\ &= [C_{(1)}^{(\rho_1, m)}]_{jk} + [C_{(2)}^{(\rho_2, m)}]_{jk}, \end{aligned}$$

which prove (3.122). Finally, by combining equations (3.122), (3.120), and (3.111) we get

$$\left\langle \prod_{j=1}^m C(f_j, g_j) \right\rangle_{\rho_1 \otimes \rho_2} = \text{pf} [C^{(\rho_1 \otimes \rho_2, m)}]. \quad (3.123)$$

Thus, $\rho_1 \otimes \rho_2$ is quasi-free with respect to \mathcal{C} . \square

3.3. Correlation matrices

In this section we define correlation matrices and present their properties. We discuss the correlation matrices of reduced states and of the tensor product of states.

Let $\mathcal{D} = (d_1, d_1^*, d_2, d_2^*, \dots, d_n, d_n^*)^t$ be a general Fermionic system of \mathcal{H} . The correlation matrix of the state ρ with respect to \mathcal{D} is defined to be the $2n \times 2n$ matrix

$$\Gamma_\rho^{\mathcal{D}} := \langle \mathcal{D} \mathcal{D}^* \rangle_\rho, \quad (3.124)$$

with the row $\mathcal{D}^* = (d_1^*, d_1, \dots, d_n^*, d_n)$ and $\langle \mathcal{D} \mathcal{D}^* \rangle_\rho$ to be understood in the sense of taking expectations of each of the operator-valued entries of the $2n \times 2n$ -matrix $\mathcal{D} \mathcal{D}^*$.

The Following Lemma describes the general form of correlation matrices.

LEMMA 3.10. *Correlation matrices are generally of the form $\Gamma = (\Gamma_{jk})_{1 \leq j, k \leq n}$ with 2×2 -matrix-valued matrix elements*

$$\Gamma_{jk} = \frac{1}{2} \left(\delta_{j,k} \mathbb{1}_2 + \begin{pmatrix} X_{jk} & Y_{jk} \\ -\overline{Y_{jk}} & -\overline{X_{jk}} \end{pmatrix} \right). \quad (3.125)$$

Here X and Y are $n \times n$ -matrices such that $X^* = X$ and $Y^* = -\bar{Y}$.

It is useful to think of correlation matrices in the block form, where a simple change of basis yields:

$$P\Gamma P^t = \frac{1}{2} \left(\mathbb{1}_{2n} + \begin{pmatrix} X & Y \\ -\bar{Y} & -\bar{X} \end{pmatrix} \right), \quad (3.126)$$

where P is the permutation matrix which maps the canonical basis vectors e_1, \dots, e_{2n} of \mathbb{C}^{2n} to $e_1, e_{n+1}, e_2, e_{n+2}, \dots, e_n, e_{2n}$. In particular, correlation matrices, as defined here, are self-adjoint.

PROOF. It is enough to prove statement (3.126). Consider the following block decomposition

$$P\Gamma P^t = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (3.127)$$

Here

$$A = (\langle d_j d_k^* \rangle_\rho)_{jk}, \quad B = (\langle d_j d_k \rangle_\rho)_{jk}, \quad C = (\langle d_j^* d_k^* \rangle_\rho)_{jk}, \quad \text{and} \quad D = (\langle d_j^* d_k \rangle_\rho)_{jk}. \quad (3.128)$$

A is Hermitian because

$$A_{jk} = \langle d_j d_k^* \rangle_\rho = \text{Tr}[d_j d_k^* \rho] = \overline{\text{Tr}[d_k d_j^* \rho]} = \overline{\langle d_k d_j^* \rangle_\rho} = \overline{A_{kj}}. \quad (3.129)$$

B is anti-symmetric, we have

$$\begin{aligned} \langle d_j d_j \rangle_\rho &= 0 \\ \langle d_j d_k \rangle_\rho &= \text{Tr}[d_j d_k \rho] = -\text{Tr}[d_k d_j \rho] = -\langle d_k d_j \rangle_\rho \quad \text{for } j \neq k. \end{aligned}$$

Then, we have $C = -\bar{B}$ because for all $1 \leq j, k \leq n$, we have

$$\langle d_j^* d_k^* \rangle_\rho = \text{Tr}[d_j^* d_k^* \rho] = \overline{\text{Tr}[d_k d_j \rho]} = -\overline{\text{Tr}[d_j d_k \rho]} = -\overline{\langle d_j d_k \rangle_\rho}. \quad (3.130)$$

Finally, we have $D = \mathbb{1} - \bar{A}$ because

$$\begin{aligned}\langle d_j^* d_k \rangle_\rho &= \overline{\text{Tr}[d_k^* d_j \rho]} = -\overline{\text{Tr}[d_j d_k^* \rho]} = -\overline{\langle d_j d_k^* \rangle_\rho}, \text{ for } j \neq k \\ \langle d_j^* d_j \rangle_\rho &= \langle \mathbb{1} - d_j d_j^* \rangle_\rho = 1 - \langle d_j d_j^* \rangle_\rho.\end{aligned}$$

Thus, $P\Gamma_\rho^{\mathcal{D}}P^t$ is symmetric and it can be written in the form

$$P\Gamma_\rho^{\mathcal{D}}P^t = \begin{pmatrix} A & B \\ -\bar{B} & \mathbb{1}_n - \bar{A} \end{pmatrix} = \frac{1}{2} \left(\mathbb{1}_{2n} + \begin{pmatrix} X & Y \\ -\bar{Y} & -\bar{X} \end{pmatrix} \right). \quad (3.131)$$

where

$$X := 2A - \mathbb{1}, \quad Y = 2B. \quad (3.132)$$

□

The following Lemma list some elementary properties of correlation matrices:

LEMMA 3.11. *Let \mathcal{D} be a Fermionic system, and $\rho \in \mathcal{B}(\mathcal{H})$ be self-adjoint with $\text{Tr}[\rho] = 1$, then $\Gamma_\rho^{W\mathcal{D}} = W\Gamma_\rho^{\mathcal{D}}W^*$ that is $\langle WDD^*W^* \rangle_\rho = W\langle DD^* \rangle_\rho W^*$, for any Bogoliubov matrix $W \in \mathbb{C}^{2n \times 2n}$.*

PROOF. The Lemma follows from the following argument

$$\begin{aligned}\text{Tr} \left[\rho (WDD^*W^*)_{ij} \right] &= \text{Tr} \left[\rho \sum_{k,l} W_{ik} (DD^*)_{kl} (W^*)_{lj} \right] \\ &= \sum_{k,l} W_{ik} \text{Tr} [\rho (DD^*)_{kl} (W^*)_{lj}] \\ &= \sum_{k,l} W_{ik} \langle (DD^*)_{kl} \rangle_\rho (W^*)_{lj}.\end{aligned} \quad (3.133)$$

□

Next we give the explicit form of the correlation matrix of the density matrices $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$, where $\{\psi_\alpha\}$ are the ONBs associated with \mathcal{D} through (3.12).

LEMMA 3.12. Let \mathcal{D} be a Fermionic system and $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$, and ψ_α are the eigenfunctions associated with \mathcal{D} by equation (3.12), then

$$\Gamma_{\rho_\alpha}^{\mathcal{D}} = \bigoplus_{k=1}^n \begin{pmatrix} \delta_{\alpha_k,0} & 0 \\ 0 & \delta_{\alpha_k,1} \end{pmatrix}, \quad (3.134)$$

where $\delta_{\alpha_k,j}$ are the Kronecker delta functions.

As an important special case of Lemma 3.12 case, the correlation matrix of the vacuum state of the d_j operators:

$$\Gamma_{|\Omega\rangle\langle\Omega|}^{\mathcal{D}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\oplus n}. \quad (3.135)$$

PROOF. First note that, using the definition (3.12) and the anti-commutation properties of the d_j , one has

$$d_j^* \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 1, \\ \pm \psi_{\alpha+e_j} & \text{if } \alpha_j = 0, \end{cases} \quad (3.136)$$

as well as

$$d_j \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 0, \\ \pm \psi_{\alpha-e_j} & \text{if } \alpha_j = 1. \end{cases} \quad (3.137)$$

For the elements $\langle d_j d_k^* \rangle_{\rho_\alpha}$ we get

$$\begin{aligned} \text{Tr}[d_j d_k^* \rho_\alpha] &= \langle \psi_\alpha, d_j d_k^* \psi_\alpha \rangle = \langle d_j^* \psi_\alpha, d_k^* \psi_\alpha \rangle \\ &= \begin{cases} 0 & \text{if } \alpha_j = 1 \text{ or } \alpha_k = 1, \\ \langle \pm \psi_{\alpha+e_j}, \pm \psi_{\alpha+e_k} \rangle = \delta_{jk} & \text{if } \alpha_j = \alpha_k = 0, \end{cases} \end{aligned} \quad (3.138)$$

Similarly, we find

$$\begin{aligned} \text{Tr}[d_j d_k \rho_\alpha] &= \langle \psi_\alpha, d_j d_k \psi_\alpha \rangle = \langle d_j^* \psi_\alpha, d_k \psi_\alpha \rangle \\ &= \begin{cases} 0 & \text{if } \alpha_j = 1 \text{ or } \alpha_k = 0, \\ \pm \langle \psi_{\alpha+e_j}, \psi_{\alpha-e_k} \rangle = 0 & \text{if } \alpha_j = 0 \text{ and } \alpha_k = 1, \end{cases} \end{aligned} \quad (3.139)$$

The general structure of correlation matrices, see Lemma 3.10 determines the remaining entries of $\Gamma_{\rho_\alpha}^{\mathcal{B}}$ and completes the proof of (5.48). \square

Next we list some more important properties of correlation matrices.

LEMMA 3.13. *Let ρ and $\tilde{\rho}$ are self-adjoint on $\mathcal{B}(\mathcal{H})$ with $\text{Tr}[\rho] = \text{Tr}[\tilde{\rho}] = 1$, and \mathcal{D} and $\tilde{\mathcal{D}}$ are Fermionic systems. If $\Gamma_\rho^{\mathcal{D}} = \Gamma_{\tilde{\rho}}^{\tilde{\mathcal{D}}}$ then:*

- (a) *There exists a unitary U such that $\Gamma_\rho^{\mathcal{D}} = \Gamma_{U^*\tilde{\rho}U}^{\mathcal{D}}$, where $\tilde{d}_j = Ud_jU^*$, for all j .*
- (b) *$\langle d^\#(f)d^\#(g) \rangle_\rho = \langle d^\#(f)d^\#(g) \rangle_{U^*\tilde{\rho}U} = \langle \tilde{d}^\#(f)\tilde{d}^\#(g) \rangle_{\tilde{\rho}}$.*
- (c) *$\langle D_jD_k \rangle_\rho = \langle D_jD_k \rangle_{U^*\tilde{\rho}U} = \langle \tilde{D}_j\tilde{D}_k \rangle_{\tilde{\rho}}$, where D_j and \tilde{D}_j are short for $D(f_j, g_j)$ and $\tilde{D}(f_j, g_j)$; respectively.*

PROOF. (a) By Lemma 3.2, there exists a unitary U , such that $\tilde{d}_j = Ud_jU^*$. Thus, for $1 \leq j, k \leq n$ we have,

$$\begin{aligned} \langle d_j^\# d_k^\# \rangle_\rho &= \langle \tilde{d}_j^\# \tilde{d}_k^\# \rangle_{\tilde{\rho}} = \langle Ud_j^\# d_k^\# U^* \rangle_{\tilde{\rho}} \\ &= \text{Tr} \left[Ud_j^\# d_k^\# U^* \tilde{\rho} \right] = \text{Tr} \left[d_j^\# d_k^\# U^* \tilde{\rho} U \right] \\ &= \langle d_j^\# d_k^\# \rangle_{U^*\tilde{\rho}U}. \end{aligned} \tag{3.140}$$

(b) We will prove $\langle d(f)d^*(g) \rangle_\rho = \langle d(f)d^*(g) \rangle_{U^*\tilde{\rho}U}$, the other choices of $\#$ are identical.

$$\begin{aligned} \langle d(f)d^*(g) \rangle_\rho &= \left\langle \left(\sum_j \bar{f}_j d_j \right) \left(\sum_k g_k d_k^* \right) \right\rangle_\rho \\ &= \sum_{j,k} \bar{f}_j g_k \langle d_j d_k^* \rangle_\rho = \sum_{j,k} \bar{f}_j g_k \langle d_j d_k^* \rangle_{U^*\tilde{\rho}U} \\ &= \langle d(f)d^*(g) \rangle_{U^*\tilde{\rho}U}. \end{aligned} \tag{3.141}$$

And since $\tilde{d}(f) = Ud(f)U^*$, then the second statement is fully proved.

(c) The third statement follows from the following argument

$$\begin{aligned} \langle D_j D_k \rangle_\rho &= \langle (d(f_j) + d^*(g_j))(d(f_k) + d^*(g_k)) \rangle_\rho \\ &= \langle d(f_j)d(f_k) \rangle_\rho + \langle d(f_j)d^*(g_k) \rangle_\rho + \langle d^*(g_j)d(f_k) \rangle_\rho + \langle d^*(g_j)d^*(g_k) \rangle_\rho \end{aligned}$$

$$\begin{aligned}
&= \langle d(f_j)d(f_k) + d(f_j)d^*(g_k) + d^*(g_j)d(f_k) + d^*(g_j)d^*(g_k) \rangle_{U^*\bar{\rho}U} \\
&= \langle D_j D_k \rangle_{U^*\bar{\rho}U}.
\end{aligned}$$

It is clear that $\tilde{D}_j = UD_jU^*$, and this will finish the proof of the Lemma. \square

3.3.1. Correlation matrix of a reduced state. The goal of this subsection is to prove the following Lemma showing the relation between the correlation matrix of a reduced state and the correlation matrix of the corresponding (full) state.

LEMMA 3.14. *Let \mathcal{C}_1 corresponds to the $c_j^{(1)}$'s. The correlation matrix $\Gamma_{\rho^1}^{\mathcal{C}_1}$ of the reduced state ρ^1 is the restriction of $\Gamma_{\rho}^{\mathcal{C}}$ to $\text{span}\{e_{2j-1}, e_{2j}, j \in \Lambda_0\}$, then*

PROOF. The Lemma follows directly from the following

$$\begin{aligned}
\langle c_j^{(1)\#} c_k^{(1)\#} \rangle_{\rho^1} &= \text{Tr} \left[c_j^{(1)\#} c_k^{(1)\#} \rho^1 \right] \\
&= \text{Tr} \left[c_j^{(1)\#} c_k^{(1)\#} \text{Tr}_{\mathcal{H}_2} \rho \right] \\
&= \text{Tr} \left[\text{Tr}_{\mathcal{H}_2} \tilde{c}_j^{\#} \tilde{c}_k^{\#} \rho \right] \\
&= \text{Tr} \left[c_j^{\#} c_k^{\#} \rho \right] \\
&= \langle c_j^{\#} c_k^{\#} \rangle_{\rho},
\end{aligned}$$

where we used in the third and the fourth steps the trace class properties in Lemma 2.3. \square

3.3.2. Correlation matrix of tensor product of states. In this subsection we show that the correlation matrix of the tensor product of states is the direct sum of the correlation matrices of these states. Lemma 3.15 below states this more precisely, where we consider the set of density matrices $\{\rho_k \in \mathcal{B}(\mathcal{H}^{(k)}), k = 1, 2, \dots, m\}$ as in Lemma 3.9.

LEMMA 3.15. *The correlation matrix of $\bigotimes_{k=1}^m \rho_k$ with respect to \mathcal{C} is given by*

$$\Gamma_{\bigotimes_{k=1}^m \rho_k}^{\mathcal{C}} = \bigoplus_{k=1}^m \Gamma_{\rho_k}^{\mathcal{C}^{(k)}}. \tag{3.142}$$

PROOF. We will prove that

$$\left(\langle c_j^\# c_r^\# \rangle_{\otimes_{k=1}^m \rho_k} \right)_{j,r=1}^n = \text{diag} \left\{ \left(\langle (c_j^{(k)})^\# (c_r^{(k)})^\# \rangle_{\rho_k} \right)_{j,r \in \Lambda^{(k)}}, k = 1, 2, \dots, m \right\}. \quad (3.143)$$

This follows directly using the general form of correlation matrices and the following argument: For $j, r \in \Lambda^{(k)}$:

$$\langle c_j^\# c_r^\# \rangle_{\otimes_{k=1}^m \rho_k} = \langle \mathbb{1}^{\otimes(k-1)} \otimes (c_j^{(k)})^\# (c_r^{(k)})^\# \otimes \mathbb{1}^{\otimes(m-k)} \rangle_{\otimes_{k=1}^m \rho_k} = \langle (c_j^{(k)})^\# (c_r^{(k)})^\# \rangle_{\rho_k}. \quad (3.144)$$

For $j \in \Lambda^{(k_1)}$, and $r \in \Lambda^{(k_2)}$ for $k_1 < k_2$: one can see that

$$\begin{aligned} c_j^\# &= (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes (\sigma^z)^{\otimes |\Lambda^{(2)}|} \otimes \dots \otimes (c_j^{(k_1)})^\# \otimes \mathbb{1}^{\otimes |\Lambda^{(k_1+1)}|} \otimes \dots \otimes \mathbb{1}^{\otimes |\Lambda^{(m)}|} \\ c_r^\# &= (\sigma^z)^{\otimes |\Lambda^{(1)}|} \otimes \dots \otimes (\sigma^z)^{\otimes |\Lambda^{(k_1)}|} \otimes \dots \otimes (c_r^{(k_2)})^\# \otimes \mathbb{1}^{\otimes |\Lambda^{(k_2+1)}|} \otimes \dots \otimes \mathbb{1}^{\otimes |\Lambda^{(m)}|}. \end{aligned}$$

Then

$$\begin{aligned} \langle c_j^\# c_r^\# \rangle_{\otimes_{k=1}^m \rho_k} &= \left\langle \mathbb{1} \otimes (c_j^{(k_1)})^\# (\sigma^z)^{\otimes |\Lambda^{(k_1)}|} \otimes (\sigma^z)^{\otimes (\sum_{k_1 < h < k_2} |\Lambda^{(h)}|)} \otimes (c_r^{(k_2)})^\# \otimes \mathbb{1} \right\rangle_{\otimes_{k=1}^m \rho_k} \\ &= \left\langle (c_j^{(k_1)})^\# (\sigma^z)^{\otimes |\Lambda^{(k_1)}|} \right\rangle_{\rho^{(k_1)}} \left(\prod_{k_1 < h < k_2} \left\langle (\sigma^z)^{\otimes |\Lambda^{(h)}|} \right\rangle_{\rho^{(h)}} \right) \langle (c_r^{(k_2)})^\# \rangle_{\rho^{(k_2)}} \\ &= 0 \end{aligned} \quad (3.145)$$

because

$$\left\langle (c_j^{(k_1)})^\# (\sigma^z)^{\otimes |\Lambda^{(k_1)}|} \right\rangle_{\rho^{(k_1)}} = \langle (c_r^{(k_2)})^\# \rangle_{\rho^{(k_2)}} = 0. \quad (3.146)$$

□

3.4. Entropy of quasi-free states

The goal of this section is to prove Theorem 3.8. It is mainly saying that the entropy of quasi-free states can be found through their correlation matrices.

3.4.1. Diagonalizing correlation matrices. In this subsection we prove that correlation matrices are diagonalizable by Bogoliubov matrices. We also prove that

correlation matrices of generic states are positive-semidefinite and has a symmetric spectrum about $\frac{1}{2}$.

LEMMA 3.16. *Let \mathcal{D} be a Fermionic system and ρ is self-adjoint on $\mathcal{B}(\mathcal{H})$, then the correlation matrix $\Gamma_\rho^{\mathcal{D}}$ is diagonalizable by a Bogoliubov matrix and has a symmetric spectrum around $\frac{1}{2}$. i.e.*

$$\Gamma_\rho^{\mathcal{D}} = W \bigoplus_{j=1}^n \begin{pmatrix} \frac{1+\lambda_j}{2} & 0 \\ 0 & \frac{1-\lambda_j}{2} \end{pmatrix} W^*$$

for some $\lambda_j \in \mathbb{R}$ and Bogoliubov matrix W .

We will prove later that in the case where ρ is a state, we have $-1 \leq \lambda_j \leq 1$, see Theorem 3.4

PROOF. It is enough to show that

$$T := 2\Gamma_\rho^{\mathcal{D}} - \mathbb{1} \tag{3.147}$$

is diagonalizable by a Bogoliubov matrix. Let us define the unitary matrix $\Omega := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}^{\oplus n}$. We will check that $-i\Omega T \Omega^* =: \Gamma$ is real anti-symmetric matrix, i.e. $\Gamma = -\Gamma^t$. This follows from proving that $P\Gamma P^t$ is anti-symmetric. Using the block form of correlation matrices in Lemma 3.10, we get

$$P\Gamma P^t = \begin{bmatrix} X & Y \\ -\bar{Y} & -\bar{X} \end{bmatrix}$$

where $X = X^*$ and $Y = -Y^t$. Thus

$$\begin{aligned} P\Gamma P^t &= \begin{bmatrix} i\frac{X-\bar{X}}{2} + \frac{Y-\bar{Y}}{2} & -\left(\frac{X+\bar{X}}{2} - \frac{Y+\bar{Y}}{2}\right) \\ \left(\frac{X+\bar{X}}{2} + \frac{Y+\bar{Y}}{2}\right) & i\frac{X-\bar{X}}{2} - \frac{Y-\bar{Y}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \text{Im}(X+Y) & \text{Re}(X-Y) \\ -\text{Re}(X+Y) & \text{Im}(X-Y) \end{bmatrix}. \end{aligned}$$

We have

$$\begin{aligned}
X^* = X &\Rightarrow (ReX + iImX)^* = ReX + iImX \\
&\Rightarrow (ReX)^t - i(ImX)^t = ReX + iImX \\
&\Rightarrow (ReX)^t = ReX, (ImX)^t = -ImX.
\end{aligned} \tag{3.148}$$

Similarly,

$$\begin{aligned}
-Y^t = Y &\Rightarrow -(ReY + iImY)^t = ReY + iImY \\
&\Rightarrow -(ReY)^t - i(ImY)^t = ReY + iImY \\
&\Rightarrow (ReY)^t = -ReY, (ImY)^t = -ImY.
\end{aligned} \tag{3.149}$$

A direct inspection using (3.148) and (3.149) implies

$$\begin{aligned}
(Im(X + Y))^t &= -(Im(X + Y)) \\
(Re(X - Y))^t &= Re(X + Y) \\
(Im(X - Y))^t &= -(Im(X - Y)) \\
-(Re(X + Y))^t &= -Re(X - Y),
\end{aligned}$$

which implies that Γ is anti-symmetric i.e., $\Gamma = -\Gamma^t$. From the spectral theory of anti-symmetric matrices, there exists an orthogonal matrix O such that

$$\Gamma = O \bigoplus_{j=1}^n \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix} O^t, \tag{3.150}$$

where $\lambda_j \geq 0$ for $j = 1, 2, \dots, n$. By diagonalizing we get

$$\Omega T \Omega^* = i\Gamma = \frac{1}{2} O \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{\oplus n} \bigoplus_{j=1}^n \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}^{\oplus n} O^t. \tag{3.151}$$

Thus

$$T = W \bigoplus_{j=1}^n \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix} W^* \text{ where } W = \frac{1}{\sqrt{2}} \Omega^* O \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{\oplus n}. \quad (3.152)$$

Then note that W is unitary because it is a multiplication of three unitary matrices and that $WJW^t = J$ using the following argument

$$\begin{aligned} WJW^t &= \frac{1}{2} \Omega^* O \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}^{\oplus n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{\oplus n} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}^{\oplus n} O^t \bar{\Omega} \\ &= \frac{1}{2} \Omega^* O O^t \bar{\Omega} \\ &= \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}^{\oplus n} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}^{\oplus n} = J. \end{aligned} \quad (3.153)$$

This proves that W is a Bogoliubov matrix. Lastly, (3.147) and (3.152) clearly show that the spectrum of the correlation matrix is symmetric about $\frac{1}{2}$. \square

Real valued correlation matrices are of great importance as we will see in the applications, and they are diagonalizable via a special explicit (orthogonal) Bogoliubov matrix:

LEMMA 3.17. *Let \mathcal{D} be a Fermionic system in \mathcal{H} and let $\rho \in \mathcal{B}(\mathcal{H})$ be self-adjoint with $\text{Tr}[\rho] = 1$ such that $\Gamma_\rho^{\mathcal{D}}$ is real valued, then $\Gamma_\rho^{\mathcal{D}}$ is diagonalizable by an (orthogonal) Bogoliubov matrix.*

PROOF. It is enough to prove the Lemma for $P\Gamma_\rho^{\mathcal{D}}P^t := \tilde{\Gamma}$, which is given by the block form

$$\tilde{\Gamma} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where $A = A^t$ and $B = -B^t$. Since

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tilde{\Gamma} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \mathbb{1}_{2n} - \tilde{\Gamma}, \quad (3.154)$$

i.e. $\tilde{\Gamma}$ is unitarily equivalent to $(\mathbb{1} - \tilde{\Gamma})$ and, in particular, $\sigma(\tilde{\Gamma}) = 1 - \sigma(\tilde{\Gamma})$. To diagonalize $\tilde{\Gamma}$ we follow the following steps: Let $S := A+B$ and $0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \leq \tilde{\lambda}_n$ be the singular values of S , i.e. the eigenvalues of $(S^*S)^{1/2}$, counted with multiplicity. Let $\tilde{\Lambda} := \text{diag}\{\tilde{\lambda}_1, \dots, \tilde{\lambda}_n\}$. The singular value decomposition of S gives orthogonal matrices U and V such that

$$USV^t = U(A+B)V^t = \tilde{\Lambda}. \quad (3.155)$$

This implies

$$\tilde{\Lambda} = \tilde{\Lambda}^t = VS^tU^t = V(A-B)U^t. \quad (3.156)$$

Let

$$W := \frac{1}{2} \begin{pmatrix} V+U & V-U \\ V-U & V+U \end{pmatrix}. \quad (3.157)$$

Then W is an orthogonal $2n \times 2n$ -matrix. This can be checked by directly verifying that $WW^t = \mathbb{1}$, or, alternatively, by noting that

$$SWS^{-1} = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}, \quad (3.158)$$

with the orthogonal matrix

$$S := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}. \quad (3.159)$$

A calculation shows that W diagonalizes $\Gamma_\rho^{\mathcal{D}}$ via

$$W\tilde{\Gamma}W^t = \begin{bmatrix} \frac{\mathbb{1}+\tilde{\Lambda}}{2} & 0 \\ 0 & \frac{\mathbb{1}-\tilde{\Lambda}}{2} \end{bmatrix}, \quad (3.160)$$

and it is not hard to check that

$$W^tJW = J, \text{ where } J = \begin{bmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{bmatrix}. \quad (3.161)$$

Thus, W is a Bogoliubov matrix. \square

The following Theorem proves the positive semi-definiteness of the correlation matrix of states.

THEOREM 3.4. *For any Fermionic system \mathcal{D} , if ρ is any state then*

$$0 \leq \Gamma_\rho^{\mathcal{D}} \leq \mathbb{1}_{2n}. \quad (3.162)$$

PROOF. By Lemma 3.16, there exists a Bogoliubov matrix W such that $W\Gamma_\rho^{\mathcal{D}}W^t = \text{diag}\{\eta_1, 1 - \eta_1, \eta_2, 1 - \eta_2, \dots, \eta_n, 1 - \eta_n\} := D_\eta$. Then by defining $\tilde{\mathcal{D}} := W\mathcal{D}$ with Fermionic operators \tilde{d}_j 's, we get $\Gamma_\rho^{\tilde{\mathcal{D}}} = D_\eta$, see Lemma 3.11. Since the spectrum of the correlation matrix is symmetric around $\frac{1}{2}$, see Lemma 3.16, then without loss of generality, we may assume that $\eta_j \geq \frac{1}{2}$, for $1 \leq j \leq n$. Thus, for $1 \leq j \leq n$ we have

$$\langle \tilde{d}_j \tilde{d}_j^* \rangle_\rho = \eta_j \Rightarrow \text{Tr}[\tilde{d}_j \tilde{d}_j^* \rho] = \eta_j \quad (3.163)$$

and since $0 \leq \tilde{d}_j \tilde{d}_j^* \leq 1$ (orthogonal projections), then $\eta_j = \text{Tr}[\tilde{d}_j \tilde{d}_j^* \rho] \leq \text{Tr}[\rho] = 1$, where we used Lemma B.2. And using the symmetry of the spectrum of $\Gamma_\rho^{\mathcal{D}}$, we get the statement of the theorem. \square

3.4.2. Correlation matrices of quasi-free states. In this subsection we show how correlation matrices of quasi-free states determine completely the state. To understand what that means, we start by diagonal states

$$\rho^{(diag)} = \bigotimes_{j=1}^n \begin{pmatrix} 1 - \eta_j & 0 \\ 0 & \eta_j \end{pmatrix}, \quad \eta_j \in \mathbb{R} \quad \text{for all } j = 1, 2, \dots, n. \quad (3.164)$$

Note that the 2^n eigenvalues of $\rho^{(diag)}$ are

$$\prod_{j=1}^n \eta_j^{\alpha_j} (1 - \eta_j)^{1 - \alpha_j}, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0, 1\}^n.$$

Theorem 3.2 gives that $\rho^{(diag)}$ are quasi-free states. A simple calculation, see below, shows that

$$\Gamma_{\rho^{(diag)}}^{\mathcal{C}} = \bigoplus_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & \mathbb{1} - \eta_j \end{pmatrix}. \quad (3.165)$$

Thus all the 2^n eigenvalues of $\rho^{(diag)}$ can be found explicitly from the $2n$ eigenvalues of its correlation matrix $\Gamma_{\rho^{(diag)}}^{\mathcal{C}}$. This also gives a formula for the entropy of $\rho^{(diag)}$:

$$\begin{aligned} -\text{Tr} [\rho^{(diag)} \log \rho^{(diag)}] &= S(\rho^{(diag)}) = \sum_{j=1}^n S \left(\begin{pmatrix} \eta_j & 0 \\ 0 & \mathbb{1} - \eta_j \end{pmatrix} \right) \\ &= -\sum_{j=1}^n \eta_j \log \eta_j - \sum_{j=1}^n (1 - \eta_j) \log(1 - \eta_j) \\ &= -\text{tr} \left[\Gamma_{\rho^{(diag)}}^{\mathcal{C}} \log \Gamma_{\rho^{(diag)}}^{\mathcal{C}} \right], \end{aligned}$$

where we used Lemma 2.2 for the entropy of product states. So far, we are able to find the correlation matrices of given states. A natural question is whether it is possible to find the state explicitly if the correlation matrix of that state is given. For example if the correlation matrix of a certain unknown state ρ with respect to a Fermionic system \mathcal{D} is found to be diagonal

$$\Gamma_{\rho}^{\mathcal{D}} = \bigoplus_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & \mathbb{1} - \eta_j \end{pmatrix} = \Gamma_{\rho^{(diag)}}^{\mathcal{C}}. \quad (3.166)$$

Can we draw any relation between ρ and $\rho^{(diag)}$? Theorem 3.5 below gives a definite answer: they are equal up to a unitary if ρ is quasi-free with respect to \mathcal{D} . In the following, we will show formula (3.165). We check the diagonal elements first, where we drop the upper-script (*diag*) for simplicity.

$$\langle c_j c_j^* \rangle_{\rho} = \text{Tr}[a_j a_j^* \rho] = \text{Tr} \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - \eta_j & 0 \\ 0 & \eta_j \end{pmatrix} \right] = \eta_j. \quad (3.167)$$

We have also

$$\langle c_j^* c_j \rangle_{\rho} = 1 - \langle c_j c_j^* \rangle_{\rho} = 1 - \eta_j. \quad (3.168)$$

For the off-diagonal elements, since the correlation matrix is Hermitian then we may assume that $j < k$ in the following argument

$$\begin{aligned} \langle c_j^\# c_k^\# \rangle_\rho &= \pm \operatorname{Tr} \left[a_j^\# \sigma_{j+1}^z \cdots \sigma_{k-1}^z a_k^\# \rho \right] \\ &= \pm \operatorname{Tr} \left[a_j^\# \begin{pmatrix} 1 - \eta_j & 0 \\ 0 & \eta_j \end{pmatrix} \otimes \begin{pmatrix} 1 - \eta_{j+1} & 0 \\ 0 & -(\eta_{j+1}) \end{pmatrix} \otimes \cdots \right. \\ &\quad \left. \cdots \otimes a_k^\# \begin{pmatrix} 1 - \eta_k & 0 \\ 0 & \eta_k \end{pmatrix} \right] \end{aligned}$$

and this is equal to zero because

$$\operatorname{Tr} \left[a_j^\# \begin{pmatrix} 1 - \eta_j & 0 \\ 0 & \eta_j \end{pmatrix} \right] = \operatorname{Tr} \left[\begin{pmatrix} 1 - \eta_k & 0 \\ 0 & \eta_k \end{pmatrix} a_k^\# \right] = 0.$$

THEOREM 3.5. *Let ρ and $\tilde{\rho}$ be states in 2^n -dimensional Hilbert spaces \mathcal{H} and $\tilde{\mathcal{H}}$, respectively, which are quasi-free with respect to the Fermionic systems \mathcal{D} and $\tilde{\mathcal{D}}$, respectively. If*

$$\Gamma_\rho^{\mathcal{D}} = \Gamma_{\tilde{\rho}}^{\tilde{\mathcal{D}}}, \quad (3.169)$$

then ρ and $\tilde{\rho}$ are unitary equivalent.

PROOF. Using notations as in Lemma 3.13 (for both \mathcal{D} and $\tilde{\mathcal{D}}$), equality of the correlation matrices gives the equality of the correlations

$$\langle D_j D_k \rangle_\rho = \langle \tilde{D}_j \tilde{D}_k \rangle_{\tilde{\rho}} \text{ for all } 1 \leq j < k \leq n. \quad (3.170)$$

Thus for any positive integer m ,

$$D^{(\rho, m)} = \tilde{D}^{(\tilde{\rho}, m)}. \quad (3.171)$$

As ρ and $\tilde{\rho}$ are quasi-free with respect to \mathcal{D} and $\tilde{\mathcal{D}}$, this implies

$$\left\langle \prod_{j=1}^m D_j \right\rangle_\rho = \operatorname{pf} [D^{(\rho, m)}] = \operatorname{pf} [\tilde{D}^{(\tilde{\rho}, m)}] = \left\langle \prod_{j=1}^m \tilde{D}_j \right\rangle_{\tilde{\rho}} = \left\langle \prod_{j=1}^m U^* D_j U \right\rangle_{\tilde{\rho}} \quad (3.172)$$

$$= \left\langle \prod_{j=1}^m D_j \right\rangle_{U\tilde{\rho}U^*},$$

where we used Lemma 3.2 which provides a unitary U such that $\tilde{d}_j = U^*d_jU$ for all $1 \leq j \leq n$. Using Lemma 3.4, this implies $\langle A \rangle_\rho = \langle A \rangle_{U\tilde{\rho}U^*}$ for any $A \in \mathcal{B}(\mathcal{H})$ and thus

$$\rho = U\tilde{\rho}U^*. \quad (3.173)$$

□

COROLLARY 3.3. *Under the same assumptions of Theorem 3.5, if W is a Bogoliubov matrix and*

$$W\Gamma_\rho^{\tilde{\mathcal{D}}}W^* = \Gamma_{\tilde{\rho}}^{\mathcal{D}},$$

then ρ and $\tilde{\rho}$ are unitary equivalent.

PROOF. By Lemma 3.11, we have

$$W\Gamma_\rho^{\tilde{\mathcal{D}}}W^* = \Gamma_\rho^{W\tilde{\mathcal{D}}} \Rightarrow \Gamma_\rho^{W\tilde{\mathcal{D}}} = \Gamma_{\tilde{\rho}}^{\mathcal{D}}.$$

Then, by Theorem 3.5, ρ and $\tilde{\rho}$ are unitary equivalent. □

THEOREM 3.6. *Let ρ be quasi-free with respect to \mathcal{D} on $\mathcal{B}(\mathcal{H})$, if $\Gamma_\rho^{\mathcal{D}}$ has eigenvalues $\{\eta_j, 1 - \eta_j, j = 1, 2, \dots, n\}$ then ρ and $\rho^{(diag)}$ given in (3.164) are unitary equivalent.*

That is, the eigenvalues of ρ can be found directly from the eigenvalues of the correlation matrix.

PROOF. Using Lemma 3.17, $\Gamma_\rho^{\mathcal{D}}$ is diagonalizable using a Bogoliubov matrix W ,

$$W\Gamma_\rho^{\mathcal{D}}W^* = \Gamma_{\rho^{(diag)}}^{\mathcal{C}} \quad (3.174)$$

where $\rho^{(diag)}$ is given by (3.164) and since ρ and $\rho^{(diag)}$ are quasi-free with respect to \mathcal{D} and \mathcal{C} ; respectively. Then by Corollary 3.3, ρ and $\rho^{(diag)}$ are unitary equivalent. □

Next we list more ties between quasi-free states and their correlation matrices.

THEOREM 3.7. Let ρ be a quasi-free state with respect to the Fermionic system \mathcal{D} on $\mathcal{B}(\mathcal{H})$, then

(a) $0 < \rho < 1$ if and only if $0 < \Gamma_\rho^{\mathcal{D}} < \mathbb{1}_{2n}$.

(b) ρ is a pure state if and only if there exists a Bogoliubov matrix W such that

$$\Gamma_\rho^{\mathcal{D}} = W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\oplus n} W^*, \quad (\text{and thus } \Gamma_\rho^{\mathcal{D}} \cdot \Gamma_\rho^{\mathcal{D}} = \Gamma_\rho^{\mathcal{D}}). \quad (3.175)$$

PROOF. (a) To prove the first statement note that ρ is a state, then using Theorem 3.4, $0 \leq \Gamma_\rho^{\mathcal{D}} \leq 1$, and it is diagonalizable via a Bogoliubov matrix W , that is,

$$W^* \Gamma_\rho^{\mathcal{D}} W = \bigoplus_{j=1}^n \begin{pmatrix} \mathbb{1} - \eta_j & 0 \\ 0 & \eta_j \end{pmatrix} = \Gamma_{\rho^{(diag)}}^{\mathcal{C}}$$

where

$$\rho^{(diag)} = \bigotimes_{j=1}^n \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}. \quad (3.176)$$

Now, since ρ and $\rho^{(diag)}$ are quasi-free with respect \mathcal{D} and \mathcal{C} ; respectively. Then ρ and $\rho^{(diag)}$ are unitary equivalent.

(b) For the second statement, if ρ be a pure state then $\rho^{(diag)}$ is a pure state as well and thus

$$W \Gamma_\rho^{\mathcal{D}} W^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\oplus n}. \quad (3.177)$$

To prove the converse, we have

$$\Gamma_\rho^{\mathcal{D}} = W^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^{\oplus n} W \Rightarrow W \Gamma_\rho^{\mathcal{D}} W^* = \Gamma_{\rho^{(pure)}}^{\mathcal{C}} \quad (3.178)$$

where $\rho^{(pure)}$ is a pure state, and using the same argument used above, ρ and $\rho^{(pure)}$ are unitary equivalent and thus ρ is a pure state. \square

We are now able to give a formula for the entropy of a quasi-free state ρ in terms of its correlation matrix.

LEMMA 3.18. *Let \mathcal{D} be a Fermionic system in \mathcal{H} , ρ is any self-adjoint operator in $\mathcal{B}(\mathcal{H})$ with $\text{Tr}[\rho] = 1$. If ρ is quasi-free with respect to \mathcal{D} then*

$$S(\rho) = -\text{Tr}[\rho \log \rho] = -\text{tr}[\Gamma_\rho^{\mathcal{D}} \log \Gamma_\rho^{\mathcal{D}}]. \quad (3.179)$$

PROOF. $\Gamma_\rho^{\mathcal{D}}$ is diagonalizable via a Bogoliubov matrix W such that

$$W\Gamma_\rho^{\mathcal{D}}W^* = \Gamma_{\rho^{(diag)}}^{\mathcal{C}}, \quad (3.180)$$

where $\rho^{(diag)}$ has the form (3.164), and since ρ and $\rho^{(diag)}$ are quasi-free with respect to \mathcal{D} and \mathcal{C} ; respectively. Then there exists a unitary U such that

$$\rho = U\rho^{(diag)}U^*. \quad (3.181)$$

Thus,

$$\begin{aligned} \text{Tr}[\rho \log \rho] &= \text{Tr}[U\rho^{(diag)}U^* \log U\rho^{(diag)}U^*] = \text{Tr}[\rho^{(diag)} \log \rho^{(diag)}] \\ &= \text{tr}\left[\Gamma_{\rho^{(diag)}}^{\mathcal{C}} \log \Gamma_{\rho^{(diag)}}^{\mathcal{C}}\right] = \text{tr}\left[W\Gamma_\rho^{\mathcal{D}}W^t \log W\Gamma_\rho^{\mathcal{D}}W^t\right] \\ &= \text{tr}\left[\Gamma_\rho^{\mathcal{D}} \log \Gamma_\rho^{\mathcal{D}}\right]. \end{aligned}$$

□

Now we can find the entanglement entropy of a quasi free state from its correlation matrix.

THEOREM 3.8. *Let ρ be a quasi-free state on $\mathcal{B}(\mathcal{H})$. The entanglement entropy of ρ with respect to the decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$ in (3.88) is given by the formula:*

$$\mathcal{E}(\rho) = -\text{Tr}[\rho^1 \log \rho^1] = -\text{tr}\left[\Gamma_{\rho^1}^{\mathcal{C}_1} \log \Gamma_{\rho^1}^{\mathcal{C}_1}\right] \quad (3.182)$$

where \mathcal{C}_1 is the local Jordan-Wigner system on Λ_0 . Moreover, the correlation matrix $\Gamma_{\rho^1}^{\mathcal{C}_1}$ of ρ^1 is the restriction of $\Gamma_\rho^{\mathcal{C}}$ to $\text{span}\{e_{2j-1}, e_{2j}, j \in \Lambda_0\}$.

PROOF. The proof follows directly from Lemmas 3.18 and 3.14, recalling that ρ^1 is quasi-free with respect to the local Jordan-Wigner Fermionic system \mathcal{C}_1 , see Theorem 3.3. □

Particle Number Transport in the Isotropic XY Chain

4.1. The model and main assumptions

For any $n \geq 1$, we consider an isotropic XY spin chain, also called the XX or the symmetric XY chain, in transversal magnetic field on $\Lambda = [1, n] := \{1, \dots, n\}$, given by the self-adjoint Hamiltonian

$$H_{\text{iso}} = - \sum_{j=1}^{n-1} \mu_j [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^n \nu_j \sigma_j^z \quad (4.1)$$

acting on the Hilbert space $\mathcal{H} = \bigotimes_{j=1}^n \mathbb{C}^2$. The parameters μ_j and ν_j describe the interaction strength and field strength, respectively. By σ_j^x , σ_j^y , and σ_j^z we denote the standard Pauli matrices, see (3.25), acting on the j -th component of the tensor product \mathcal{H} . Our main interest is in the behavior of random systems, and so we often think of the parameters, indicated above, as the first n components of sequences of real-valued random variables indexed by $j \in \mathbb{N}$. Assumptions on the random parameters will be implicit, as we will state our results under the condition of *eigencorrelator localization* of the *effective one-particle Hamiltonian* A associated with H_{iso} , given by (4.10) below. This will be understood as the existence of a non-increasing function $F : [0, \infty) \rightarrow (0, \infty)$, of which we will require that it vanishes sufficiently fast as the argument tends to ∞ , and such that

$$\mathbb{E} \left(\sup_{|g| \leq 1} |\langle e_j, g(A) e_k \rangle| \right) \leq F(|j - k|), \quad (4.2)$$

uniformly in $n \in \mathbb{N}$ and $1 \leq j, k \leq n$. Here, the supremum is taken over arbitrary Borel functions $g : \mathbb{R} \rightarrow \mathbb{C}$ with modulus pointwise bounded by 1 and $g(A)$ is defined via the functional calculus of hermitian matrices.

Typical examples of the function F in the right hand side of (4.2) are

$$F(r) = Ce^{-r\eta} \quad \text{and} \quad F(r) = C/(1+r)^\beta, \quad (4.3)$$

where η , β and C are positive constants. In particular, if $\mu_j = \mu \in \mathbb{R} \setminus \{0\}$ and ν_j are i.i.d. random variables from absolutely continuous distribution ρ with bounded compactly supported density, then A is the Anderson model on the finite interval Λ , and it is known that for some $C < \infty$ and η , (4.2) is satisfied with $F(r) = Ce^{-r\eta}$.

4.2. Reduction to the single particle Hamiltonian

The importance of the isotropic XY chain as a model in the theory of quantum spin systems goes back to the work of Lieb, Schultz and Mattis [LSM61], where it was shown that the XY chain with constant coefficients (and initially without magnetic field) is an exactly solvable model. Their argument proceeds by using the Jordan-Wigner transform to reduce the XY chain to a free Fermion system. For the last half century this has turned the XY chain into a canonical toy model for quantum spin systems, frequently used as a first example to illustrate new concepts.

It was understood that the methods of Lieb, Schultz and Mattis can be extended to include magnetic fields.

Below we present this reduction. Pauli matrices can be written in terms of raising and lowering operators,

$$\sigma^x = a_j + a_j^*, \quad \sigma^y = i(a_j - a_j^*), \quad \sigma^z = 2a_j^*a_j - \mathbb{1}. \quad (4.4)$$

Using these identities one verifies easily

$$\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y = 2(a_j^* a_{j+1} + a_{j+1}^* a_j), \quad j = 1, \dots, n-1. \quad (4.5)$$

Similarly,

$$\sigma_j^x \sigma_{j+1}^x - \sigma_j^y \sigma_{j+1}^y = 2(a_j a_{j+1} + a_{j+1}^* a_j^*). \quad (4.6)$$

We can express H_{iso} in terms of the operators a_j as

$$H_{\text{iso}} = -2 \sum_{j=1}^{n-1} \mu_j [a_j a_{j+1}^* + a_{j+1} a_j^*] - \sum_{j=1}^n \nu_j (2a_j^* a_j - \mathbb{1}). \quad (4.7)$$

With c_j as in (3.34) and using

$$a_j a_{j+1} = c_j c_{j+1}, \quad a_{j+1}^* a_j^* = c_{j+1}^* c_j^*, \quad a_{j+1}^* a_j = -c_{j+1} c_j, \quad a_j^* a_j = c_j^* c_j, \quad (4.8)$$

one finds that the Hamiltonian H_{iso} can be rewritten in terms of the c_j operators as

$$\begin{aligned} H_{\text{iso}} &= 2 \sum_{j=1}^{n-1} \mu_j (c_j^* c_{j+1} + c_{j+1}^* c_j) - 2 \sum_{j=1}^n \nu_j c_j^* c_j + \sum_{j=1}^n \nu_j \mathbb{1} \\ &= 2c^* A c + E_0 \mathbb{1}. \end{aligned} \quad (4.9)$$

Here $c := (c_1, \dots, c_n)^t$, $c^* = (c_1^*, \dots, c_n^*)$, $E_0 := \sum_j \nu_j$ and A is the symmetric Jacobi matrix

$$A = (A_{jk})_{j,k=1}^n := \begin{pmatrix} -\nu_1 & \mu_1 & & & \\ \mu_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \mu_{n-1} & \\ & & \mu_{n-1} & -\nu_n & \end{pmatrix}. \quad (4.10)$$

There is an orthogonal matrix U such that

$$U A U^t = \tilde{\Lambda} = \text{diag}\{\tilde{\lambda}_j, j = 1, 2, \dots, n\}, \quad (4.11)$$

where $\{\tilde{\lambda}_j\}_j$, are the eigenvalues of A . Define $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_n)^t$ by

$$\tilde{b} = U c. \quad (4.12)$$

We give two arguments to show that the $\{\tilde{b}_j\}_j$ satisfy the CAR. In the first argument, we refer to the machinery in Chapter 3, where (4.12) can be written as

$$\begin{pmatrix} \tilde{b} \\ \tilde{b}^* \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} c \\ c^* \end{pmatrix},$$

and one can see that this is a Bogoliubov transformation, see (3.43). Without referring to Chapter 3, it is also easy to see that the $\{\tilde{b}_j\}_j$ satisfy the CAR using the linearity of $\{\cdot, \cdot\}$ and orthogonality of U , as follows. First, the orthogonal transformation (4.12) gives

$$\tilde{b}_j = \sum_{\ell=1}^n U_{j\ell} c_\ell, \quad \tilde{b}_j^* = \sum_{r=1}^n U_{jr} c_r^*.$$

Then, the bi-linearity of $\{\cdot, \cdot\}$ gives

$$\{\tilde{b}_j, \tilde{b}_k\} = \sum_{\ell, r=1}^n U_{j\ell} U_{kr} \{c_\ell, c_r\} = 0.$$

Finally, the orthogonality of U gives

$$\{\tilde{b}_j, \tilde{b}_k^*\} = \sum_{\ell, r=1}^n U_{j\ell} U_{kr} \{c_\ell, c_r^*\} = \sum_{\ell, r=1}^n U_{j\ell} U_{kr} \delta_{\ell, r} \mathbb{1} = \sum_{\ell=1}^n U_{j\ell} U_{k\ell} \mathbb{1} = \delta_{j, k} \mathbb{1}.$$

Then by substituting (4.12) in (4.9),

$$\begin{aligned} H_{\text{iso}} &= 2c^* U^t \tilde{\Lambda} U c + E_0 \mathbb{1} = 2\tilde{b}^* \tilde{\Lambda} \tilde{b} + E_0 \mathbb{1} \\ &= 2 \sum_{j=1}^n \tilde{\lambda}_j \tilde{b}_j^* \tilde{b}_j + E_0 \mathbb{1}. \end{aligned} \tag{4.13}$$

Thus H_{iso} has been written in the form of a *free Fermion system*.

4.3. Particle number operator

In this section we present the particle number operator, and show how the particle number is a conserved quantity by the isotropic XY chain (4.1).

The *particle number operator* is defined as follows

$$\mathcal{N} = \sum_{j=1}^n a_j^* a_j, \quad \text{where } a^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.14}$$

are the basic raising and lowering operators. First, note that

$$a^* a |e_\uparrow\rangle = |e_\uparrow\rangle \langle e_\uparrow | e_\uparrow\rangle = |e_\uparrow\rangle. \tag{4.15}$$

And if we denote the up-down-spin product basis states by

$$e_\alpha := e_{\alpha_1} \otimes \dots \otimes e_{\alpha_n}, \quad \alpha \in \{\uparrow, \downarrow\}^n, \quad (4.16)$$

then one can see that \mathcal{N} counts the number of up-spins in e_α :

$$\mathcal{N}e_\alpha = N_\alpha e_\alpha, \quad \text{where } N_\alpha = |\{j : \alpha_j = \uparrow\}|. \quad (4.17)$$

Thus, \mathcal{N} counts the number of up-spins (particles) and for this, \mathcal{N} got its name.

One of the most important characteristics of the isotropic XY Hamiltonian H_{iso} is that it commutes with the particle number operator, i.e. $[H_{\text{iso}}, \mathcal{N}] = 0$, this follows from the following argument:

$$[H_{\text{iso}}, \mathcal{N}] = - \sum_{j=1}^{n-1} \sum_{k=1}^n \mu_j [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y, a_k^* a_k] - \sum_{j=1}^n \sum_{k=1}^n \nu_j [\sigma_j^z, a_k^* a_k]. \quad (4.18)$$

A direct calculation gives that

$$[\sigma_j^z, a_k^* a_k] = 0, \quad \text{for all } j \text{ and } k.$$

In the first double sum, one can see that it simplifies to

$$\begin{aligned} &= - \sum_{j=1}^{n-1} \mu_j ([\sigma_j^x \sigma_{j+1}^x, a_j^* a_j] + [\sigma_j^y \sigma_{j+1}^y, a_{j+1}^* a_{j+1}]) - \\ &\qquad \sum_{j=1}^{n-1} \mu_j ([\sigma_j^x \sigma_{j+1}^x, a_{j+1}^* a_{j+1}] + [\sigma_j^y \sigma_{j+1}^y, a_j^* a_j]). \end{aligned} \quad (4.19)$$

A calculation shows that each term in the previous two big sums is zero, and this proves that $[H_{\text{iso}}, \mathcal{N}] = 0$.

The fact that H_{iso} commutes with the number operators implies that

$$[e^{-itH_{\text{iso}}}, \mathcal{N}] = 0.$$

Then for the dynamics $(e_\alpha)_t = e^{-itH_{\text{iso}}}e_\alpha$, see Section 2.2,

$$\mathcal{N}(e_\alpha)_t = \mathcal{N}e^{-itH_{\text{iso}}}e_\alpha = e^{-itH_{\text{iso}}}\mathcal{N}e_\alpha = N_\alpha (e_\alpha)_t. \quad (4.20)$$

Also we have, in sense of (2.3) and (2.4):

$$\langle \mathcal{N} \rangle_{\rho_t} = \langle \mathcal{N} \rangle_\rho \quad (4.21)$$

for any state ρ and all times t , meaning that the number of up-spins (particles) is conserved in time. This is called the *conservation of particle number*.

4.4. Weak particle number transport

The goal of this section is to present and prove our main result about the particle number transport, Theorem 4.1.

For a subset $S \subset \Lambda$ we also define the local number operator, counting the number of up-spins in S , as

$$\mathcal{N}_S := \sum_{j \in S} a_j^* a_j. \quad (4.22)$$

As initial states ρ we will consider a product states given by arbitrary density profiles (diagonal states),

$$\rho = \bigotimes_{j=1}^n \rho_j, \quad \rho_j = \begin{pmatrix} \eta_j & 0 \\ 0 & 1 - \eta_j \end{pmatrix}, \quad 0 \leq \eta_j \leq 1, \quad j = 1, \dots, n. \quad (4.23)$$

For these initial states ρ we are interested in how the expectation of the observable \mathcal{N}_S changes under the Heisenberg evolution of H_{iso} . The latter is given by $\tau_t(A) = e^{itH_{\text{iso}}} A e^{-itH_{\text{iso}}}$ for any $A \in \mathcal{B}(\mathcal{H})$, the bounded linear operators over \mathcal{H} , while the Schrödinger evolution of the state ρ is $\rho_t = e^{-itH_{\text{iso}}}\rho e^{itH_{\text{iso}}}$. Recall that we are denoting the expectation of an observable A in the state ρ by $\langle A \rangle_\rho := \text{Tr}[A\rho]$, we thus have to analyze the quantity

$$\langle \mathcal{N}_S \rangle_{\rho_t} = \text{Tr}[\mathcal{N}_S \rho_t] = \text{Tr}[\mathcal{N}_S e^{-itH_{\text{iso}}}\rho e^{itH_{\text{iso}}}] \quad (4.24)$$

$$\begin{aligned}
&= \text{Tr} [e^{itH_{\text{iso}}}\mathcal{N}_S e^{-itH_{\text{iso}}}\rho] = \text{Tr} [\tau_t(\mathcal{N}_S)\rho] \\
&= \langle \tau_t(\mathcal{N}_S) \rangle_\rho.
\end{aligned}$$

$\mathbb{E}(\cdot)$ denoting the disorder average, we will prove the following bound on the number of up-spins (particles) in S . (Part (a₂) and (4.25) below are adapted from [SW16a].)

THEOREM 4.1. *Consider H_{iso} and assume eigencorrelator localization (4.2) for the effective Hamiltonian A . Then*

$$\begin{aligned}
(a_1) \quad \mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) &\leq \sum_{j \in S} \sum_{k=1}^n \eta_k F(|j-k|). \\
(a_2) \quad \mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) &\leq \langle \mathcal{N}_K \rangle_\rho + \sum_{j \in S, k \in K^c} F(|j-k|), \text{ for arbitrary pairs of subsets} \\
&S \subseteq K \subseteq \Lambda.
\end{aligned}$$

Moreover,

$$\mathbb{E} \left(\sup_t |\langle \mathcal{N}_S \rangle_{\rho_t} - \langle \mathcal{N}_S \rangle_\rho| \right) \leq 2 \sum_{j \in S, k \in S^c} F(|j-k|). \quad (4.25)$$

If, for simplicity, we think of all η_j to be zeros or ones, where ρ becomes a pure state associated with an up-down spin configuration, then parts (a₁) and (a₂) from the theorem give an upper bound of the number of up-spins created in S in time. With the same proof, we can similarly bound the number of down-spins (holes) in (a₁): If

$$\tilde{\mathcal{N}}_S := \sum_{j \in S} a_j a_j^* = |S| \mathbb{1} - \mathcal{N}_S,$$

then

$$\mathbb{E} \left(\sup_t \langle \tilde{\mathcal{N}}_S \rangle_{\rho_t} \right) \leq \sum_{j \in S} \sum_{k=1}^n (1 - \eta_k) F(|j-k|). \quad (4.26)$$

Statement (4.25) gives a bound for the maximum number of the newly created up-spins in S . In particular, if F is decaying exponentially or polynomially as in (4.3) then

$$\mathbb{E} \left(\sup_t |\langle \mathcal{N}_S \rangle_{\rho_t} - \langle \mathcal{N}_S \rangle_\rho| \right) \leq C \quad (4.27)$$

where $C < \infty$ is independent of the sizes of S and chain Λ . This can be seen from the fact that

$$\sum_{j \in S, k \in S^c} F(|j - k|) < 2 \sum_{j=1}^{\infty} jF(j).$$

This means that the total number of the newly created particles in S is bounded, regardless of the situation outside S . Meaning that if the initial state ρ is void only in S , i.e.

$$\rho = |\psi\rangle\langle\psi|, \text{ where } \psi = (e_{\downarrow})^{\otimes S} \otimes (e_{\uparrow})^{\otimes \Lambda \setminus S}, \quad (4.28)$$

then a weak particle transport can be concluded.

An interesting special case of Theorem 4.1 arises when starting with all up-spins in a subinterval $\Lambda_0 \subset \Lambda$ and all down-spins in $\Lambda \setminus \Lambda_0$, i.e. for the pure state

$$\rho = |\varphi\rangle\langle\varphi|, \text{ with } \varphi = (e_{\uparrow})^{\otimes \Lambda_0} \otimes (e_{\downarrow})^{\otimes \Lambda \setminus \Lambda_0}, \quad (4.29)$$

meaning $\eta_k = 1$ for $k \in \Lambda_0$ and $\eta_k = 0$ otherwise in (4.23). If $S \subset \Lambda \setminus \Lambda_0$, then parts (a₁) and (a₂) with $K = \Lambda \setminus \Lambda_0$ in the latter implies

$$\mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) \leq \sum_{j \in S} \sum_{k \in \Lambda_0} F(|j - k|) \leq 2 \sum_{j=d(S, \Lambda_0)}^{\infty} jF(|j|), \quad (4.30)$$

where $d(A, B) = \min\{|a - b| : a \in A, b \in B\}$ denotes the distance between two sets.

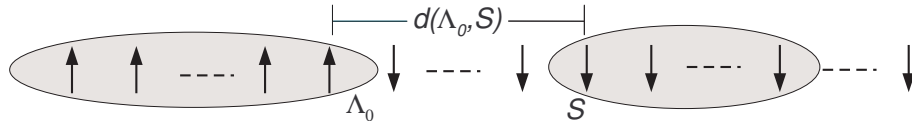


FIGURE 4.1. Subregions Λ_0 and S over the initial state φ , here Λ_0 occupies the left end of the chain.

This is a bound on the expectation of the number of up-spins which penetrate from Λ_0 into S . If F has sufficiently rapid decay, then the right hand side of (4.30) is not only finite, uniformly in the sizes of Λ and Λ_0 , but decaying for growing distance $d(S, \Lambda_0)$. More precisely, the following Corollary lists two cases:

COROLLARY 4.1. Assume the eigencorrelator localization (4.2) for the effective Hamiltonian A and the initial state ρ given in (4.29). Then

(a) If $F(r) = Ce^{-\eta r}$ for some C and $\eta < \infty$ then

$$\mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) \leq \tilde{C} e^{-\frac{\eta}{2} d(S, \Lambda_0)} \quad (4.31)$$

for some $\tilde{C} < \infty$.

(b) If $F(r) = C/(1+r)^\beta$ for some $C < \infty$ and $\beta > 2$ then

$$\mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) \leq \frac{C_\beta}{d(S, \Lambda_0)^{\beta-2}}, \quad (4.32)$$

for some $C_\beta < \infty$.

Here \tilde{C} and C_β are independent of the sizes of S , Λ_0 , and Λ .

4.5. Proof of main results

The goal of this section is to prove Theorem 4.1 and Corollary 4.1.

PROOF OF THEOREM 4.1. Using that $a_j^* a_j = c_j^* c_j$, for all $1 \leq j \leq n$, the operator \mathcal{N}_S can be written as

$$\mathcal{N}_S = \sum_{j \in S} a_j^* a_j = \sum_{j \in S} c_j^* c_j. \quad (4.33)$$

Thus, the quantity of interest here is

$$\langle \mathcal{N}_S \rangle_{\rho_t} = \sum_{j \in S} \langle c_j^* c_j \rangle_{\rho_t} = \sum_{j \in S} \langle \tau_t(c_j^* c_j) \rangle_\rho = \sum_{j \in S} \langle \tau_t(c_j^*) \tau_t(c_j) \rangle_\rho \quad (4.34)$$

where we used in the last step that the Heisenberg evolution of a product of two observables A and B is the product of their Heisenberg evolutions, i.e.

$$\tau_t(AB) = e^{itH_{\text{iso}}} A B e^{-itH_{\text{iso}}} = e^{itH_{\text{iso}}} A e^{-itH_{\text{iso}}} e^{itH_{\text{iso}}} B e^{-itH_{\text{iso}}} = \tau_t(A) \tau_t(B). \quad (4.35)$$

Next we will find an explicit formulas for $\tau_t(c_j^*)$ and $\tau_t(c_j)$. First, for any $1 \leq k \leq n$, observe that

$$[H_{\text{iso}}, \tilde{b}_k] = 2 \sum_{j=1}^n \tilde{\lambda}_j [\tilde{b}_j^* \tilde{b}_j, \tilde{b}_k] = -2\tilde{\lambda}_k \tilde{b}_k. \quad (4.36)$$

Next, we will show that

$$\tau_t(\tilde{b}_k) = e^{-2it\tilde{\lambda}_k} \tilde{b}_k. \quad (4.37)$$

First, define the functions

$$h_k(t) = e^{itH_{\text{iso}}} \tilde{b}_k e^{-itH_{\text{iso}}}. \quad (4.38)$$

It is easy to see that

$$h'_k(t) = ie^{itH_{\text{iso}}} [H_{\text{iso}}, \tilde{b}_k] e^{-itH_{\text{iso}}} = -2i\tilde{\lambda}_k h_k(t), \quad (4.39)$$

and so

$$h_k(t) = e^{-2it\tilde{\lambda}_k} h_k(0) = e^{-2it\tilde{\lambda}_k} \tilde{b}_k. \quad (4.40)$$

In vector form, this can be expressed as $\tau_t(\tilde{b}) = e^{-2it\tilde{\Lambda}} \tilde{b}$, where $\tilde{\Lambda}$ is given in (4.11).

This implies

$$\tau_t(c) = \tau_t(U^t \tilde{b}) = U^t \tau_t(\tilde{b}) = U^t e^{-2it\tilde{\Lambda}} \tilde{b} = U^t e^{-2it\tilde{\Lambda}} U c = e^{-2itA} c \quad (4.41)$$

where we used (4.11) and the change of variables (4.12). This means that

$$\tau_t(c_j) = \sum_{k=1}^n (e^{-2itA})_{jk} c_k, \quad \tau_t(c_j^*) = \sum_{\ell=1}^n \overline{(e^{-2itA})_{j\ell}} c_\ell^*. \quad (4.42)$$

Thus we get

$$\begin{aligned} \langle \tau_t(c_j^*) \tau_t(c_j) \rangle_\rho &= \sum_{k,\ell=1}^n \overline{(e^{-2itA})_{j\ell}} (e^{-2itA})_{jk} \langle c_\ell^* c_k \rangle_\rho \\ &= \sum_{k=1}^n \overline{(e^{-2itA})_{jk}} (e^{-2itA})_{jk} \eta_k \end{aligned} \quad (4.43)$$

where we used in the second step, with the specific initial density matrix $\rho = \bigotimes_j \rho_j$ given in (4.23), that

$$\langle C_\ell^* C_k \rangle_\rho = \delta_{\ell,k} \eta_k. \quad (4.44)$$

By substituting (4.43) in (4.34), we find

$$\langle \mathcal{N}_S \rangle_{\rho_t} = \sum_{j \in S} \sum_{k=1}^n \eta_k |(e^{2itA})_{jk}|^2. \quad (4.45)$$

Thus we can use eigencorrelator localization (4.2) to find that

$$\begin{aligned} \mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) &\leq \sum_{j \in S} \sum_{k=1}^n \eta_k \mathbb{E} \left(\sup_t |(e^{2itA})_{jk}|^2 \right) \\ &\leq \sum_{j \in S} \sum_{k=1}^n \eta_k \mathbb{E} \left(\sup_t |(e^{2itA})_{jk}| \right) \\ &\leq \sum_{j \in S} \sum_{k=1}^n \eta_k F(|j-k|), \end{aligned} \quad (4.46)$$

where we used that $|(e^{2itA})_{jk}|^2 \leq |(e^{2itA})_{jk}|$ because e^{2itA} is a unitary matrix, and this proves statement (a₁).

To prove (a₂), we start by noticing that

$$\langle \mathcal{N}_K \rangle_\rho = \sum_{j \in K} \eta_j = \text{tr}[\chi_K \eta] \quad (4.47)$$

where χ_K is the restriction operator to the set K , and $\eta := \text{diag}\{\eta_j, j = 1, 2, \dots, n\}$.

Moreover, $\langle \mathcal{N}_S \rangle_{\rho_t}$ in (4.45) can be written as

$$\langle \mathcal{N}_S \rangle_{\rho_t} = \text{tr}[\chi_S U_t^* \eta U_t] \quad (4.48)$$

where $U_t := e^{2itA}$. Thus

$$\begin{aligned} \langle \mathcal{N}_S \rangle_{\rho_t} &= \text{tr}[\chi_S U_t^* \eta U_t] \\ &= \text{tr}[\chi_S U_t^* \chi_K \eta U_t] + \text{tr}[\chi_S U_t^* \chi_{K^c} \eta U_t] \\ &\leq \text{tr}[U_t^* \chi_K \eta U_t] + \text{tr}[\chi_S U_t^* \chi_{K^c} \eta U_t] \end{aligned}$$

$$\begin{aligned}
&\leq \operatorname{tr}[\chi_K \eta] + \|\chi_S U_t^* \chi_{K^c}\|_1 \|\eta U_t\| \\
&\leq \langle \mathcal{N}_K \rangle_\rho + \sum_{j \in S, k \in K^c} |(U_t^*)_{jk}|.
\end{aligned}$$

In the third-to-fourth step, we used that for any trace class operators A and B , we have $|\operatorname{tr}[AB]| \leq \|A\|_1 \|B\|$, see Lemma B.3 for a proof. In the fourth-to-last step, we used that $\|\eta U_t\| \leq 1$, and the fact that the 1-norm can be bounded by the sum of the absolute value of the matrix elements, see Lemma B.4 for a proof. Finally, by taking the sup over t then averaging we get statement (a_1) .

To prove (4.25), we proceed as follows

$$\begin{aligned}
|\langle \mathcal{N}_S \rangle_{\rho_t} - \langle \mathcal{N}_S \rangle_\rho| &= |\operatorname{tr}[\chi_S U_t^* \eta U_t] - \operatorname{tr}[\chi_S \eta]| \\
&= |\operatorname{tr}[\chi_S U_t^* (\chi_S + \chi_{S^c}) \eta U_t] - \operatorname{tr}[(\chi_S + \chi_{S^c}) U_t^* \chi_S \eta U_t]| \\
&= |\operatorname{tr}[\chi_S U_t^* \chi_{S^c} \eta U_t] - \operatorname{tr}[\chi_{S^c} U_t^* \chi_S \eta U_t]| \\
&\leq \|\chi_S U_t^* \chi_{S^c}\|_1 \|\eta U_t\| + \|\chi_{S^c} U_t^* \chi_S\|_1 \|\eta U_t\| \\
&\leq \sum_{j \in S, k \in S^c} |(U_t^*)_{jk}| + \sum_{j \in S^c, k \in S} |(U_t^*)_{jk}|
\end{aligned}$$

Applying the sup over t and averaging yield (4.25). \square

Next we prove Corollary 4.1, where we use the bound in (4.30) as a starting point.

PROOF OF COROLLARY 4.1. (a) For the case of power decay $F(r) = C/(1+r)^\beta$, using bound (4.30) we get

$$\begin{aligned}
\mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) &\leq 2C \sum_{j=d(S, \Lambda_0)}^{\infty} \frac{j}{(1+j)^\beta} \leq 2C \sum_{j=d(S, \Lambda_0)}^{\infty} \frac{1}{(1+j)^{\beta-1}} \\
&\leq 2C \int_{d(S, \Lambda_0)-1}^{\infty} (1+x)^{1-\beta} dx \\
&= \frac{C_\beta}{d(S, \Lambda_0)^{\beta-2}}, \quad (\beta > 2)
\end{aligned}$$

where $C_\beta = 2C(\beta-2)^{-1}$. In the third step we used the integral bound for the series.

(b) Exponential decay of F leads to exponential decay of $\mathbb{E}(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t})$ in $d(S, \Lambda_0)$, as follows.

$$\begin{aligned}
\mathbb{E} \left(\sup_t \langle \mathcal{N}_S \rangle_{\rho_t} \right) &\leq C \sum_{j \in S} \sum_{k \in \Lambda_0} e^{-\eta|j-k|} \\
&= C \sum_{j \in S} \sum_{k \in \Lambda_0} e^{-\frac{\eta}{2}|j-k|} e^{-\frac{\eta}{2}|j-k|} \\
&\leq C e^{-\frac{\eta}{2}d(S, \Lambda_0)} \sum_{j \in S} \sum_{k \in \Lambda_0} e^{-\frac{\eta}{2}|j-k|} \\
&\leq C e^{-\frac{\eta}{2}d(S, \Lambda_0)} \sum_{j \in \Lambda \setminus \Lambda_0} \sum_{k \in \Lambda_0} e^{-\frac{\eta}{2}|j-k|} \\
&\leq 2C e^{-\frac{\eta}{2}d(S, \Lambda_0)} \sum_{j=1}^{\infty} j e^{-\frac{\eta}{2}j} \\
&\leq \tilde{C} e^{-\frac{\eta}{2}d(S, \Lambda_0)}.
\end{aligned}$$

Here $\tilde{C} = 2C \sum_{j=1}^{\infty} j e^{-\frac{\eta}{2}j}$, and the convergent series can be estimated using the corresponding integral bound. \square

Note here that the proof of Theorem 4.1 does not extend to the case of the anisotropic XY chain, as can be expected physically: In the anisotropic case we do not have particle number conservation. Thus up-spins can be created by local properties of the dynamics, for example by simply flipping a down-spin into an up-spin. This can not be prevented by being in the regime of many-body localization, which is the main physical mechanism exploited in our proof.

Entanglement Dynamics of Disordered XY Chains

5.1. The model and main assumptions

We consider an anisotropic XY spin chain in transversal magnetic field given by the self-adjoint Hamiltonian

$$H = H_{[1,n]} = - \sum_{j=1}^{n-1} \mu_j [(1 + \gamma_j) \sigma_j^x \sigma_{j+1}^x + (1 - \gamma_j) \sigma_j^y \sigma_{j+1}^y] - \sum_{j=1}^n \nu_j \sigma_j^z \quad (5.1)$$

in $\mathcal{H} = \bigotimes_{j=1}^n \mathbb{C}^2$, where $[1, n] := \{1, \dots, n\}$ for an arbitrary positive integer n . As in Chapter 4, σ_j^x , σ_j^y , and σ_j^z denote the standard Pauli matrices acting on the j -th component of the tensor product. The parameters μ_j , γ_j , and ν_j describe the interaction strength, anisotropy and field strength, respectively, and we will think of them as the first n components of sequences of real-valued random variables indexed by $j \in \mathbb{N}$. To be more precise, our standing assumptions will be that all three sequences $\{\mu_j\}_{j \in \mathbb{N}}$, $\{\gamma_j\}_{j \in \mathbb{N}}$, and $\{\nu_j\}_{j \in \mathbb{N}}$ are i.i.d., also independent from one another, and that they have distributions of bounded support.

Stronger assumptions on the random parameters will be given later, as we will assume that the eigencorrelator localization of the effective one-particle Hamiltonian M , given by (5.5) below, is satisfied. This in analogue of (4.2) reads

$$\mathbb{E} \left(\sup_{|g| \leq 1} \|g(M)_{jk}\| \right) \leq F(|j - k|), \quad (5.2)$$

uniformly in $n \in \mathbb{N}$ and $1 \leq j, k \leq n$. But here, we view $g(M)$ as an $n \times n$ -matrix with 2×2 -matrix-valued entries and thus $\|\cdot\|$ on the left of (5.2) is a norm on the 2×2 -matrices, which, for definiteness, we choose to be the Euclidean matrix norm.

And as in Chapter 4, $F : [0, \infty) \rightarrow (0, \infty)$ vanishes sufficiently fast with typical examples as in (4.3).

5.2. Reduction to the single particle Hamiltonian

The Jordan-Wigner transform used in Chapter 4 can be extended to the anisotropic case with variable coefficients μ_j , γ_j , and ν_j , see [HSS12]. And the diagonalization of H can be reduced to the diagonalization of the effective Hamiltonian M .

Below we present this reduction. Tracing the first steps from the isotropic case (Section 4.2), statement (4.7) reads

$$H = -2 \sum_{j=1}^{n-1} \mu_j [a_j a_{j+1}^* + a_{j+1} a_j^* + \gamma_j (a_j a_{j+1} + a_{j+1}^* a_j^*)] - \sum_{j=1}^n \nu_j (2a_j^* a_j - \mathbb{1}). \quad (5.3)$$

Using the identities (4.8), the XY Hamiltonian in its general anisotropic form (5.1) can be written in terms of the c_j operators as

$$\begin{aligned} H &= -2 \sum_{j=1}^{n-1} \mu_j [c_j c_{j+1}^* + c_{j+1} c_j^* + \gamma_j (c_j c_{j+1} + c_{j+1}^* c_j^*)] - \sum_{j=1}^n \nu_j (2c_j^* c_j - \mathbb{1}) \\ &= - \sum_{j=1}^{n-1} \mu_j [c_j c_{j+1}^* - c_{j+1}^* c_j + c_{j+1} c_j^* - c_j^* c_{j+1} + \\ &\quad \gamma_j (c_j c_{j+1} - c_{j+1} c_j + c_{j+1}^* c_j^* - c_j^* c_{j+1}^*)] - \sum_{j=1}^n \nu_j (c_j^* c_j - c_j c_j^*) \\ &= \mathcal{C}^* M \mathcal{C}, \end{aligned} \quad (5.4)$$

where \mathcal{C} is the Jordan-Wigner Fermionic system (see Section 3.1.1)

$$\mathcal{C} = (c_1, c_1^*, c_2, c_2^*, \dots, c_n, c_n^*)^t, \quad \mathcal{C}^* = (c_1^*, c_1, c_2^*, c_2, \dots, c_n^*, c_n),$$

and M is the effective one-particle Hamiltonian which can be written in form of the symmetric scalar $2n \times 2n$ -matrix M given by

$$M := \begin{pmatrix} -\nu_1 \sigma^z & \mu_1 S(\gamma_1) & & & \\ \mu_1 S(\gamma_1)^t & -\nu_2 \sigma^z & \ddots & & \\ & \ddots & \ddots & \mu_{n-1} S(\gamma_{n-1}) & \\ & & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z \end{pmatrix}, \quad (5.5)$$

where

$$S(\gamma) = \begin{pmatrix} 1 & \gamma \\ -\gamma & -1 \end{pmatrix}. \quad (5.6)$$

Next, we diagonalize the Hamiltonian H . It is more comfortable to deal with the block form of M given in (5.7) below. This is done using the permutation matrix P which maps the canonical basis vectors e_1, \dots, e_{2n} of \mathbb{C}^{2n} to $e_1, e_{n+1}, e_2, e_{n+2}, \dots, e_n, e_{2n}$,

$$PMP^t =: \tilde{M} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix}, \quad (5.7)$$

where

$$A = \begin{pmatrix} -\nu_1 & \mu_1 & & & \\ \mu_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \mu_{n-1} & \\ & & & \mu_{n-1} & -\nu_n \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \gamma_1 \mu_1 & & & \\ -\gamma_1 \mu_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \gamma_{n-1} \mu_{n-1} & \\ & & & -\gamma_{n-1} \mu_{n-1} & 0 \end{pmatrix}. \quad (5.8)$$

We have $A^* = A^t = A$ and $B^* = B^t = -B$, and thus

$$\tilde{M}^* = \begin{pmatrix} A^* & -B^* \\ B^* & -A^* \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix} = \tilde{M}. \quad (5.9)$$

Transforming with the unitary

$$\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tilde{M} \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = -\tilde{M}, \quad (5.10)$$

we see that \tilde{M} is unitarily equivalent to $-\tilde{M}$, and thus, $\sigma(\tilde{M}) = -\sigma(\tilde{M})$, i.e. \tilde{M} has spectrum symmetric about zero.

Let $S := A + B$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the singular values of S , i.e. the eigenvalues of $(S^*S)^{1/2}$, counted with multiplicity. Let $\Lambda := \text{diag}\{\lambda_1, \dots, \lambda_n\}$. The singular value decomposition of S gives orthogonal matrices U and V such that

$$USV^t = U(A + B)V^t = \Lambda. \quad (5.11)$$

This implies

$$\Lambda = \Lambda^t = VS^tU^t = V(A - B)U^t. \quad (5.12)$$

Let

$$\hat{W} := \frac{1}{2} \begin{pmatrix} V + U & V - U \\ V - U & V + U \end{pmatrix}. \quad (5.13)$$

Then \hat{W} is an orthogonal $2n \times 2n$ -matrix. This can be checked by directly verifying that $\hat{W}\hat{W}^t = \mathbb{1}$, or, alternatively, by noting that \hat{W} is unitarily equivalent to an orthogonal matrix,

$$S\hat{W}S^{-1} = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix}, \text{ with the orthogonal matrix } S := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix}.$$

Calculations show that

$$\begin{aligned} \hat{W}\tilde{M}\hat{W}^t &= \frac{1}{4} \begin{pmatrix} 2U(A + B)V^t + 2V(A - B)U^t & 2U(A + B)V^t - 2V(A - B)U^t \\ 2V(A - B)U^t - 2U(A + B)V^t & -2U(A + B)V^t - 2V(A - B)U^t \end{pmatrix} \\ &= \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \end{aligned}$$

where (5.11) and (5.12) are used in the last step.

This means

$$\tilde{M} = \hat{W}^t \begin{pmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{pmatrix} \hat{W}, \quad W M W^t = \bigoplus_{j=1}^n \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix}, \quad (5.14)$$

where $W = P^t \hat{W} P$ is Bogoliubov. Let

$$\mathcal{B} := W\mathcal{C}. \quad (5.15)$$

By Lemma 3.1 this is a Fermionic system and

$$\begin{aligned} H &= \mathcal{C}^* M \mathcal{C} = \mathcal{B}^* W M W^t \mathcal{B} \\ &= \mathcal{B}^* \bigoplus_{j=1}^n \begin{pmatrix} \lambda_j & 0 \\ 0 & -\lambda_j \end{pmatrix} \mathcal{B} \\ &= \sum_{j=1}^n \lambda_j (b_j^* b_j - b_j b_j^*) \end{aligned}$$

using the CAR, we get

$$H = 2 \sum_{j=1}^n \lambda_j b_j^* b_j - E_0 \mathbb{1}, \quad \text{where } E_0 = \sum_{j=1}^n \lambda_j. \quad (5.16)$$

Thus H has been written in the form of a free Fermion system. Let Ω be the vacuum vector of the b_j and

$$\psi_\alpha = (b_1^*)^{\alpha_1} \dots (b_n^*)^{\alpha_n} \Omega, \quad \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{0, 1\}^n \quad (5.17)$$

the orthonormal basis of \mathcal{H} associated with \mathcal{B} . Using (3.13), the ψ_α form a complete set of eigenvectors for H with corresponding eigenvalues $2 \sum_{j:\alpha_j=1} \lambda_j - E_0$, so that the spectrum of H is

$$\sigma(H) = \left\{ E_\alpha := 2 \sum_{j:\alpha_j=1} \lambda_j - E_0; \alpha \in \{0, 1\}^n \right\}. \quad (5.18)$$

Note that

$$\sigma(H) \text{ is simple} \Rightarrow \sigma(M) \text{ is simple.} \quad (5.19)$$

This can be seen by proving that the contrapositive is true: Suppose that there exists an $\ell \in \{1, 2, \dots, n-1\}$ such that $\lambda_\ell = \lambda_{\ell+1}$, then let

$$\alpha = (\alpha_1, \dots, \alpha_{\ell-1}, 1, 0, \alpha_{\ell+1}, \dots, \alpha_n), \quad \beta = (\alpha_1, \dots, \alpha_{\ell-1}, 0, 1, \alpha_{\ell+1}, \dots, \alpha_n)$$

and note that $\alpha \neq \beta$ but $E_\alpha = E_\beta$.

5.3. Problem setup and results

The set-up is as before, where we consider the general anisotropic XY chain Hamiltonian $H = H_{[1,n]}$, as in (5.1), on volume $\Lambda = [1, n]$. We will need to assume that, for every $n \in \mathbb{N}$,

$$H_{[1,n]} \text{ almost surely has simple spectrum.} \quad (5.20)$$

Let $\Lambda_0 = [a, b]$ be an arbitrary subinterval of Λ and consider the bi-partite decomposition

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \text{with} \quad \mathcal{H}_1 = \bigotimes_{j \in \Lambda_0} \mathbb{C}^2 \quad \text{and} \quad \mathcal{H}_2 = \bigotimes_{j \in \Lambda \setminus \Lambda_0} \mathbb{C}^2. \quad (5.21)$$

By $\rho^1 = \text{Tr}_{\mathcal{H}_2} \rho$ we denote the partial trace of a state ρ in \mathcal{H} found by tracing out the variables in $\Lambda \setminus \Lambda_0$, see Section 2.5, and $\mathcal{E}(\rho) = \mathcal{S}(\rho^1)$ the entanglement entropy of ρ with respect to the decomposition of Λ into Λ_0 and $\Lambda \setminus \Lambda_0$, see Section 2.6.

We will consider the Schrödinger time evolution $\rho_t = e^{-itH} \rho e^{itH}$ of suitable initial states ρ and study how their bipartite entanglement $\mathcal{E}(\rho_t)$ with respect to this decomposition grows in time. Towards this goal we can handle initial states which are products of *any finite number* of eigenstates of restrictions of the XY Hamiltonian (5.1) to subsystems.

More precisely, let $1 = r_0 < r_1 < \dots < r_m = n$ be integers and, for each $1 \leq k < m$, set $\Lambda_k = [r_{k-1}, r_k - 1]$, while $\Lambda_m = [r_{m-1}, r_m]$. Thus Λ is a disjoint union of the intervals Λ_k with $k = 1, \dots, m$.

For $1 \leq k \leq m$, consider the restrictions H_{Λ_k} of the XY Hamiltonian $H = H_\Lambda$ to Λ_k , defined similar to (5.1),

$$H_{\Lambda_k} = - \sum_{j \in \Lambda_k: j+1 \in \Lambda_k} \mu_j [(1 + \gamma_j) \sigma_j^x \sigma_{j+1}^x + (1 - \gamma_j) \sigma_j^y \sigma_{j+1}^y] - \sum_{j \in \Lambda_k} \nu_j \sigma_j^z \quad (5.22)$$

which are self-adjoint operators on $\mathcal{H}_{\Lambda_k} = \bigotimes_{j \in \Lambda_k} \mathbb{C}^2$.

For each k let ψ_k be a normalized eigenstate of H_{Λ_k} and let $\rho_{\psi_k} = |\psi_k\rangle\langle\psi_k|$. We choose the initial state

$$\rho = \bigotimes_{k=1}^m \rho_{\psi_k}. \quad (5.23)$$

We can now state our second main result, an area law for the entanglement dynamics of product states of the form (5.23):

THEOREM 5.1. *Assume that the anisotropic random XY chain (5.1) has almost sure simple spectrum (5.20) and that M satisfies eigencorrelator localization (5.2) with $F(r) = C/(1+r)^\beta$ for some $\beta > 6$. Consider an initial state ρ as given by (5.23) and its Schrödinger evolution $\rho_t = e^{-iHt} \rho e^{iHt}$ under the full XY chain Hamiltonian H .*

Then there exists $C < \infty$ such that

$$\mathbb{E} \left(\sup_{t, \{\psi_k\}_{k=1,2,\dots,m}} \mathcal{E}(\rho_t) \right) \leq C \quad (5.24)$$

for all n , m , a and b with $1 \leq a \leq b \leq n$, $1 \leq m \leq n$ and all decompositions $\Lambda_1, \dots, \Lambda_m$ of $\Lambda = [1, n]$. In (5.28) the supremum is taken over all $t \in \mathbb{R}$ and all normalized eigenfunctions ψ_k of H_{Λ_k} , $k = 1, \dots, m$.

In general, we do not require the decomposition $\Lambda_1, \dots, \Lambda_m$ to be compatible with the decomposition into Λ_0 and $\Lambda \setminus \Lambda_0$. However, if Λ_0 is chosen to be a union of adjacent Λ_k , say of $\Lambda_r, \dots, \Lambda_s$, see Figure 5.1 as an example, then the initial state is a product state (not entangled) with respect to $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\mathcal{E}(\rho_{t=0}) = \mathcal{E}(\rho) = \mathcal{S}(\rho_{\psi_r} \otimes \dots \otimes \rho_{\psi_s}) = 0, \quad (5.25)$$

as $\rho_{\psi_r} \otimes \dots \otimes \rho_{\psi_s}$ is a pure state. For $t > 0$, ρ_t is generally not a product state and thus $(\rho_t)^1$ not a pure state, so that $\mathcal{E}(\rho_t)$ will be strictly positive, with upper bound given by the “volume law” $\log \dim \mathcal{H}_1 = |\Lambda_0| \log 2$, see Lemma 2.1(b).

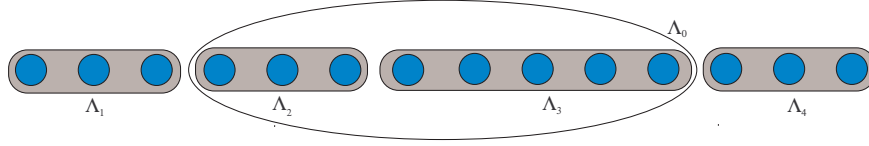


FIGURE 5.1. An example with $m = 4$, the subinterval Λ_0 is chosen to be $\Lambda_2 \cup \Lambda_3$.

The “area law” (5.28), giving an upper bound for the entanglement dynamics proportional to the surface area of the subsystem Λ_0 (with surface given by its two endpoints), is uniform not only in time $t \in \mathbb{R}$, the size of the system Λ , and the subsystem Λ_0 , but also applies uniformly to *all* possible products of eigenstates of H_{Λ_k} for $k = 1, 2, \dots, m$, irrespective of their energy. As a special case one could choose a product of ground states of the H_{Λ_k} , but our result goes far beyond this and reflects the fact that the random XY chain is a model of a *fully many-body localized* quantum system.

We comment on the two extreme cases $m = n$ and $m = 1$:

(i) In the extreme case when $m = n$ we have $\Lambda_k = \{k\}$ and $H_{\Lambda_k} = -\nu_k \sigma_k^z$, so that all interaction terms in the XY chain have been removed. For each k we have $\sigma(H_{\Lambda_k}) = \{\nu_k, -\nu_k\}$, which is almost surely simple if the distribution of the ν_k does not have an atom at 0. The eigenvectors are $|e_\uparrow\rangle$ and $|e_\downarrow\rangle$, so that the corresponding initial states for Theorem 5.1 become $\rho = |e_\alpha\rangle\langle e_\alpha|$ with arbitrary up-down-spin configurations e_α given by (4.16).

Then Theorem 5.1 gives that eigencorrelator localization of the effective Hamiltonian M implies an area law for the Schrödinger evolution $e^{-itH}|e_\alpha\rangle\langle e_\alpha|e^{itH}$ of arbitrary up-down-spin configurations. In fact, in this case the proof allows to slightly weaken the required decaying rate of the eigencorrelators:

COROLLARY 5.1. *Assume that H is almost surely simple and that M has localized eigencorrelators with $F(r) = C/(1+r)^\beta$ for some $\beta > 4$.*

Then there exists $C < \infty$ such that

$$\mathbb{E} \left(\sup_{t,\alpha} \mathcal{E}(e^{-itH}|e_\alpha\rangle\langle e_\alpha|e^{itH}) \right) \leq C \quad (5.26)$$

for all n , a and b with $1 \leq a \leq b \leq n$.

(ii) In the other extreme case when $m = 1$. In this case (5.23) simply means that $\rho = \rho_\psi$ for a normalized eigenstate ψ of H . As these states are stationary under the time-evolution, $\rho_t = \rho$, Theorem 5.1, respectively its proof, yields the following Corollary:

COROLLARY 5.2 ([ARS15]). *Assume that H is almost surely simple and that M has localized eigencorrelators with $F(r) = C/(1+r)^\beta$ for some $\beta > 2$. There exists $C < \infty$ such that*

$$\mathbb{E} \left(\sup_{\psi} \mathcal{E}(|\psi\rangle\langle\psi|) \right) \leq C, \quad (5.27)$$

where the supremum is taken over all eigenvectors ψ of H .

Again, we will prove this corollary at the end of Section 5.6.

5.4. Higher moments

Theorem 5.1 and Corollaries 5.1, 5.2 are showing that the entanglement entropy of a large class of states in the XY chain is bounded after disorder average. A natural question to ask is whether we can extend the results and prove an almost sure bound. One way to tackle this is by finding (if possible) a uniform bound for the p -moments of entanglement entropy of these states, in this case we conclude that the entanglement entropy is essentially bounded.

THEOREM 5.2. *Assume that the anisotropic random XY chain (5.1) has almost sure simple spectrum (5.20) and that M satisfies eigencorrelator localization (5.2) with*

$F(r) = C/(1+r)^\beta$ for some $\beta > 6p$ and p is a positive integer. Consider an initial state ρ as given by (5.23) and its Schrödinger evolution $\rho_t = e^{-iHt}\rho e^{iHt}$ under the full XY chain Hamiltonian H .

Then there exists $C < \infty$ such that

$$[\mathbb{E}(X^p)]^{\frac{1}{p}} \leq C_p, \quad \text{where } X := \sup_{t, \{\psi_k\}_{k=1,2,\dots,m}} \mathcal{E}(\rho_t) \quad (5.28)$$

for all n, m, a and b with $1 \leq a \leq b \leq n$, $1 \leq m \leq n$ and all decompositions $\Lambda_1, \dots, \Lambda_m$ of $\Lambda = [1, n]$.

In the isotropic case with constant $\mu_j = \mu$, the eigencorrelator localization (5.2) holds with an exponential decaying function F . i.e. $F(r) = Ce^{-\eta r}$, see Section 5.5 below. In this case, an explicit bound can be found.

COROLLARY 5.3. *In the isotropic case, assume that the effective single particle Hamiltonian M satisfies eigencorrelator localization (5.2) with $F(r) = Ce^{-\eta r}$. Consider an initial state ρ as given by (5.23) and its Schrödinger evolution $\rho_t = e^{-iHt}\rho e^{iHt}$ under the full XY chain.*

Let p be a positive integer, then there exists $C < \infty$ such that

$$[\mathbb{E}(X^p)]^{\frac{1}{p}} \leq 4 \log 2 \cdot (C \cdot D')^{1/p} \cdot \left(\frac{1}{1 - e^{-\frac{\eta}{12p}}} \right)^2 \quad (5.29)$$

for all n, m, a and b with $1 \leq a \leq b \leq n$, $1 \leq m \leq n$ and all decompositions $\Lambda_1, \dots, \Lambda_m$ of $\Lambda = [1, n]$. In (5.29),

$$D' = \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{6}|z|} \right) \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{3}|z|} \right).$$

One can see from (5.29) that $(\mathbb{E}(X^p))^{1/p}$ diverges as $\mathcal{O}(p^2)$ for $p \rightarrow \infty$. Thus no almost sure bounds can be concluded with the method used here.

5.5. Applications

Here we discuss some concrete examples, i.e. assumptions on the random variables μ_j , γ_j and ν_j which guarantee (5.20) and, most importantly, one-particle localization of M in the form (5.2).

We will focus on the case of random magnetic field, where the corresponding one-particle localization properties are best understood, i.e. we choose $\mu_j = \mu \in \mathbb{R} \setminus \{0\}$ and $\gamma_j = \gamma \in \mathbb{R}$ to be constant, while the ν_j are i.i.d. random variables with compactly supported distribution ρ . Assuming, moreover, that ρ is absolutely continuous $d\rho(\nu) = h(\nu)d\nu$, it follows from Proposition A.1 in [ARS15] that (5.20) is satisfied as well.

The localization property (5.2) is known for the following cases.

(i) Let $\gamma = 0$, so that

$$H = -\mu \sum_j (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) - \sum_j \nu_j \sigma_j^z \quad (5.30)$$

is the isotropic XY chain in random field. Then all 2×2 -matrix-entries of M are diagonal and thus M decomposes into $A \oplus (-A)$, where

$$A = \begin{pmatrix} -\nu_1 & \mu & & & \\ \mu & \ddots & \ddots & & \\ & \ddots & \ddots & \mu & \\ & & & \mu & -\nu_n \end{pmatrix} \quad (5.31)$$

is the Anderson model on the finite interval Λ . If the density h is bounded and compactly supported, then it is known that, for some $C < \infty$ and $\eta > 0$,

$$\mathbb{E} \left(\sup_{|g| \leq 1} |g(A)_{jk}| \right) \leq C e^{-\eta|j-k|}, \quad (5.32)$$

see e.g. [Sto11]. This readily implies (5.2) with exponential decay in $|j - k|$.

(ii) For the anisotropic case $\gamma \neq 0$, which does not reduce to the Anderson model, localization properties of one-particle Hamiltonians given by block Jacobi matrices of the form (5.5) have been studied more recently.

A result by Elgart, Shamis and Sodin [ESS14] covers the large disorder case: If the i.i.d. random variables $\nu_j^{(0)}$, $j \in \mathbb{N}$, have bounded compactly supported density, and if $\nu_j = \lambda \nu_j^{(0)}$ for $\lambda > 0$ sufficiently large, then (5.2) holds with exponential decay in $|j - k|$. This is done in [ESS14] through an adaptation of the fractional moments method to a class of random block operators which includes our model (5.5).

If the magnetic field is strong enough to create a spectral gap for M around energy $E = 0$ (uniform in the volume n and the disorder), then we can apply a result from [CS15]: Suppose that for some $C > |\mu|$ it holds that either $\nu_j \geq C$ for all $j \in \mathbb{N}$ or $\nu_j \leq -C$ for all $j \in \mathbb{N}$, resulting in the spectral gap $(-(C - |\mu|), C - |\mu|)$ for M . Then for every $\xi < 1$ there is $C = C(\xi)$ and $\eta = \eta(\xi) > 0$ such that

$$\mathbb{E} \left(\sup_{|g| \leq 1} \|g(M)_{jk}\| \right) \leq C e^{-\eta|j-k|^\xi}. \quad (5.33)$$

This is essentially what is shown in the proof of Theorem 7.2 of [CS15], while the results there are only stated in terms of dynamical localization for M , i.e. for the functions $g_t(x) = e^{-itx}$ (but the argument covers general $|g| \leq 1$).

5.6. Proofs of results

In this section we prove our results about the entanglement dynamics, Theorem 5.1 and Corollaries and 5.1 and 5.2. Our goal is to analyze the dynamical entanglement entropy $\mathcal{E}(\rho_t)$ of the state with initial density matrix $\rho = \bigotimes_{k=1}^m \rho_{\psi_k}$. We start by proving that ρ_t given by (recall)

$$\rho_t = e^{-2itH} \bigotimes_{k=1}^m \rho_{\psi_k} e^{2itH} \quad (5.34)$$

is quasi-free. This will be done in three steps: First, we will show that $\{\rho_{\psi_k}\}_{k=1}^m$ are quasi-free with respect to the corresponding local Jordan-Wigner Fermionic system.

Then using Lemma 3.9, the tensor product of (local) quasi-free states is quasi-free. Finally, using Lemma 5.2 below, we get that the time evolution under the full XY chain of an arbitrary quasi-free state is quasi-free.

The importance of this lies in the fact that quasi-free states ρ are completely characterised by their correlation matrices $\Gamma_\rho^{\mathcal{C}}$ defined in Section 3.3. Moreover, the von Neumann entropy of a reduced state ρ^1 is determined by a restriction of the correlation matrix. Here we restate Theorem 3.8 for the reader's convenience, as a Lemma.

LEMMA 5.1. *Let ρ be a quasi-free state on $\mathcal{B}(\mathcal{H})$. The entanglement entropy of ρ with respect to the decomposition $\mathcal{H}_1 \otimes \mathcal{H}_2$ in (5.21) is given by the formula:*

$$\mathcal{E}(\rho) = -\text{Tr} [\rho^1 \log \rho^1] = -\text{tr} \left[\Gamma_{\rho^1}^{\mathcal{C}_1} \log \Gamma_{\rho^1}^{\mathcal{C}_1} \right] \quad (5.35)$$

where \mathcal{C}_1 is the local Jordan-Wigner system on Λ_0 . Moreover, the correlation matrix $\Gamma_{\rho^1}^{\mathcal{C}_1}$ of ρ^1 is the restriction of $\Gamma_\rho^{\mathcal{C}}$ to $\text{span}\{e_{2j-1}, e_{2j}, j \in \Lambda_0\}$.

We note that, for the case of the ground state of the XY chain in constant magnetic field, the identity (5.35) was first given in [VLRK03]. There, together with an exact expression for the correlation matrix in the thermodynamic limit $\Lambda \rightarrow \mathbb{Z}$, (5.35) was used to numerically predict the dependence of the ground state entanglement on the size of the subsystem. These predictions, again for constant field and in the thermodynamic limit, were later rigorously proven, see [IJK05, IMM08] for most general results and additional references.

We proceed with proving that every eigenstates ρ_{ψ_k} of the XY chain \mathcal{H}_{Λ_k} defined on $\mathcal{B}(\mathcal{H}_{\Lambda_k})$ for all $k = 1, 2, \dots, m$ is quasi-free state with respect to the local Jordan-Wigner Fermionic system

$$\mathcal{C}^{(k)} = \left(c_1^{(k)}, (c_1^{(k)})^*, \dots, c_{|\Lambda_k|}^{(k)}, (c_{|\Lambda_k|}^{(k)})^* \right)^t.$$

As recalling from the end of Section 5.2 that

$$\psi_\alpha = (b_1^*)^{\alpha_1} (b_2^*)^{\alpha_2} \dots (b_n^*)^{\alpha_n} \Omega, \quad \alpha \in \{0, 1\}^n$$

are the eigenfunctions of the Hamiltonian H . Lemma 3.2 states that $\rho_\alpha = |\psi_\alpha\rangle\langle\psi_\alpha|$ are quasi-free with respect to the Fermionic system \mathcal{B} . The Bogoliubov transformation between \mathcal{B} and \mathcal{C} implies that $\{\rho_\alpha\}_\alpha$ are quasi-free with respect to the Jordan-Wigner Fermionic system \mathcal{C} .

The same argument applies to restrictions of the XY chain to the subinterval Λ_k of Λ . H_{Λ_k} reduces to

$$H_{\Lambda_k} = \mathcal{C}^{(k)} M_k \mathcal{C}^{(k)},$$

where M_k is the $2|\Lambda_k| \times 2|\Lambda_k|$ effective single particle Hamiltonian of H_{Λ_k} , it is the restriction of the block-Jacobi matrix (5.5) to Λ_k . Let $\{\lambda_j^{(k)}, -\lambda_j^{(k)}, k = 1, \dots, |\Lambda_k|\}$ be the eigenvalues of M_k where $\lambda_j^{(k)} \geq 0$ for all j . Following the same steps as in Section 5.2, a Bogoliubov transformation $\mathcal{B}^{(k)}$ of the local Jordan-Wigner system $\mathcal{C}^{(k)}$ will allow to write H_{Λ_k} in the form of a free Fermion system

$$H_{\Lambda_k} = 2 \sum_{j=1}^{|\Lambda_k|} \lambda_j^{(k)} (b_j^{(k)})^* b_j^{(k)} - \left(\sum_{j=1}^{|\Lambda_k|} \lambda_j^{(k)} \right) \mathbb{1}. \quad (5.36)$$

Thus, the eigenfunctions of H_{Λ_k} are

$$\psi_\alpha^{(k)} := \left((b_1^{(k)})^* \right)^{\alpha_1} \dots \left((b_1^{(k)})^* \right)^{\alpha_{|\Lambda_k|}} \Omega^{(k)},$$

where $\Omega^{(k)}$ is the vacuum vector of the $b_j^{(k)}$ Fermionic operators. As in the full XY chain, the eigenstates $|\psi_{\alpha^{(k)}}\rangle\langle\psi_{\alpha^{(k)}}|$ are quasi-free with respect to the local Jordan-Wigner Fermionic system $\mathcal{C}^{(k)}$. We know that H has almost surely simple spectrum. Since γ_j , μ_j , and ν_j are i.i.d. then by translation invariance the restriction to Λ_k , H_{Λ_k} has simple spectrum almost surely. This means that any given eigenfunction ψ_k of H_{Λ_k} corresponds, up to a phase, to one of the Fermion eigenfunctions $\psi_{\alpha^{(k)}}$. i.e. there

exists an $\alpha^{(k)} \in \{0, 1\}^{|\Lambda_k|}$ such that

$$\rho_{\psi_k} = \rho_{\alpha^{(k)}} = |\psi_{\alpha^{(k)}}\rangle\langle\psi_{\alpha^{(k)}}|.$$

And thus, any local eigenstate ρ_{ψ_k} is quasi-free with respect to the corresponding local Jordan-Wigner Fermionic system.

LEMMA 5.2. *If $\tilde{\rho}$ is an arbitrary quasi-free state then its Schrödinger evolution $\tilde{\rho}_t$ under the XY chain H is quasi-free for all $t \in \mathbb{R}$.*

PROOF. For $f, g : \Lambda \rightarrow \mathbb{C}$, define $B(f, g)$ as in (3.20), then using (4.37) we get

$$\begin{aligned} e^{itH} B(f, g) e^{-itH} &= e^{itH} \left(\sum_{j=1}^n \bar{f}_j b_j + \sum_{k=1}^n g_k b_k^* \right) e^{-itH} \\ &= \sum_{j=1}^n e^{-2it\lambda_j} \bar{f}_j b_j + \sum_{k=1}^n e^{2it\lambda_k} g_k b_k^* \\ &= B(e^{2it\Lambda} f, e^{2it\Lambda} g) =: B(f_t, g_t), \end{aligned} \quad (5.37)$$

where $\Lambda = \text{diag}\{\lambda_j, j = 1, \dots, n\}$. Now for $f_j, g_j : \Lambda \rightarrow \mathbb{C}$, and any positive integer m ,

$$\begin{aligned} \left\langle \prod_{j=1}^m B(f_j, g_j) \right\rangle_{\tilde{\rho}_t} &= \left\langle \prod_{j=1}^m e^{-itH} B(f_j, g_j) e^{itH} \right\rangle_{\tilde{\rho}} \\ &= \left\langle \prod_{j=1}^m B(f_{j,t}, g_{j,t}) \right\rangle_{\tilde{\rho}} \\ &= \text{pf} \left[B_t^{(\tilde{\rho}, m)} \right], \end{aligned} \quad (5.38)$$

where we used the cyclicity of the trace and (5.37), and $B_t^{(\tilde{\rho}, m)}$ is an anti-symmetric $m \times m$ matrix, whose jk -th element (for $1 \leq j < k \leq m$) is

$$[B_t^{(\tilde{\rho}, m)}]_{jk} = \langle B(f_{k,t}, g_{k,t}) B(f_{j,t}, g_{j,t}) \rangle_{\tilde{\rho}} = \langle B(f_k, g_k) B(f_j, g_j) \rangle_{\tilde{\rho}_t} =: [B^{(\tilde{\rho}_t, m)}]_{jk} \quad (5.39)$$

and extended appropriately by antisymmetry. Thus,

$$\left\langle \prod_{j=1}^m B(f_j, g_j) \right\rangle_{\tilde{\rho}_t} = \text{pf} [B^{(\tilde{\rho}_t, m)}]. \quad (5.40)$$

Thus $\tilde{\rho}_t$ is quasi-free with respect to \mathcal{B} and thus with respect to \mathcal{C} . \square

This completes the last step in showing that ρ_t is quasi-free.

In order to apply Lemma 5.1 to ρ_t , we see that we must investigate the correlation matrix $\Gamma_{\rho_t}^{\mathcal{C}}$. To see how this correlation matrix evolves in time, we use the following lemma.

LEMMA 5.3. *If $\tilde{\rho}$ be a density matrix on $\mathcal{B}(\mathcal{H})$ and $\tilde{\rho}_t = e^{-itH} \tilde{\rho} e^{itH}$ is its Schrödinger evolution under the XY chain. Then the correlation matrix of $\tilde{\rho}_t$ is given by*

$$\Gamma_{\tilde{\rho}_t}^{\mathcal{C}} = e^{-2itM} \Gamma_{\tilde{\rho}}^{\mathcal{C}} e^{2itM}. \quad (5.41)$$

PROOF. From the free Fermion form (in terms of the b -operators) of the Hamiltonian generating the dynamics, see (5.16), and following the same argument in finding the dynamics of \tilde{b}_j under H_{iso} , in (4.37), one finds that

$$\tau_t(b_j) = e^{-2it\lambda_j} b_j \quad \text{and} \quad \tau_t(b_j^*) = e^{2it\lambda_j} b_j^*. \quad (5.42)$$

In vector form this can be expressed as

$$\tau_t(\mathcal{B}) = \bigoplus_{j=1}^n \begin{pmatrix} e^{-2it\lambda_j} & 0 \\ 0 & e^{2it\lambda_j} \end{pmatrix} \mathcal{B}, \quad (5.43)$$

which implies

$$\begin{aligned} \tau_t(\mathcal{C}) &= \tau_t(W^* \mathcal{B}) = W^* \tau_t(\mathcal{B}) \\ &= W^* \bigoplus_{j=1}^n \begin{pmatrix} e^{-2it\lambda_j} & 0 \\ 0 & e^{2it\lambda_j} \end{pmatrix} \mathcal{B} \end{aligned}$$

$$\begin{aligned}
&= W^* \bigoplus_{j=1}^n \begin{pmatrix} e^{-2it\lambda_j} & 0 \\ 0 & e^{2it\lambda_j} \end{pmatrix} WC \\
&= e^{-2itM} \mathcal{C}.
\end{aligned}$$

As a result,

$$\Gamma_{\rho_t}^{\mathcal{C}} = \langle \tau_t(\mathcal{C}\mathcal{C}^*) \rangle_{\rho} = \langle e^{-2itM} \mathcal{C}\mathcal{C}^* e^{2itM} \rangle_{\rho} \quad (5.44)$$

and then (5.41) follows using that $\langle \cdot \rangle_{\rho}$ is linear. \square

Thus, using Lemma 3.15 giving that the correlation matrix of a tensor product of states is the direct sum of the correlation matrices corresponding to each of the states, we get that

$$\Gamma_{\rho_t}^{\mathcal{C}} = e^{-2itM} \bigoplus_{k=1}^m \Gamma_{\rho_{\psi^{(k)}}}^{\mathcal{C}^{(k)}} e^{2itM}. \quad (5.45)$$

Next we find the correlation matrices $\Gamma_{\rho_{\psi^{(k)}}}^{\mathcal{C}^{(k)}}$ in terms of the effective one particle Hamiltonian.

LEMMA 5.4. *Assume that M has simple spectrum. Then, for each $\alpha \in \{0, 1\}^n$ and the corresponding eigenfunctions ψ_{α} , given in (5.17), of H . The correlation matrix with respect to the Jordan-Wigner Fermionic system \mathcal{C} of the eigenstates $\rho_{\alpha} = |\psi_{\alpha}\rangle\langle\psi_{\alpha}|$ is given by*

$$\Gamma_{\rho_{\alpha}}^{\mathcal{C}} = \chi_{\Delta_{\alpha}}(M), \quad (5.46)$$

i.e. the spectral projection for the effective one particle Hamiltonian M onto the set

$$\Delta_{\alpha} := \{\lambda_j : \alpha_j = 0\} \cup \{-\lambda_j : \alpha_j = 1\}. \quad (5.47)$$

Here $\{\lambda_j, -\lambda_j : j = 1, 2, \dots, n\}$ are the eigenvalues of M .

Note that, in particular, $\rho_0 = |\Omega\rangle\langle\Omega|$ is the ground state projection for H and

$$\Gamma_{\rho_0}^{\mathcal{C}} = \chi_{\{\lambda_1, \dots, \lambda_n\}}(M) = \chi_{(0, \infty)}(M).$$

The difference between $\Gamma_{\rho_0}^{\mathcal{C}}$ and $\Gamma_{\rho_\alpha}^{\mathcal{C}}$ is that λ_j is replaced by $-\lambda_j$ for each site j in which a “particle is created” by b_j^* in (5.17).

PROOF. Note that using (5.19) the non-degeneracy of the spectrum of M follows from the non-degeneracy of the spectrum of H . The first step in the proof is to recall that, see Lemma 3.12

$$\Gamma_{\rho_\alpha}^{\mathcal{B}} = \bigoplus_{k=1}^n \begin{pmatrix} \delta_{\alpha_k,0} & 0 \\ 0 & \delta_{\alpha_k,1} \end{pmatrix}, \quad (5.48)$$

where \mathcal{B} is the Fermionic system (5.15). As $\mathcal{C} = W^t \mathcal{B}$, by Lemma 3.1,

$$\Gamma_{\rho_\alpha}^{\mathcal{C}} = W^t \Gamma_{\rho_\alpha}^{\mathcal{B}} W = W^t \bigoplus_{k=1}^n \begin{pmatrix} \delta_{\alpha_k,0} & 0 \\ 0 & \delta_{\alpha_k,1} \end{pmatrix} W. \quad (5.49)$$

Now, since the eigenvalues of M are simple, i.e. $-\lambda_n < \dots < -\lambda_1 < 0 < \lambda_1 < \dots < \lambda_n$, then

$$\bigoplus_{k=1}^n \begin{pmatrix} \delta_{\alpha_k,0} & 0 \\ 0 & \delta_{\alpha_k,1} \end{pmatrix}$$

is the spectral projection for

$$\bigoplus_{k=1}^n \begin{pmatrix} \lambda_k & 0 \\ 0 & -\lambda_k \end{pmatrix}$$

onto Δ_α . Therefore (5.46) follows from (5.14). \square

Now, by noting that the simplicity assumption (5.20) requires that, almost surely, $H_{[1,n]}$ is simple for *all* $n \in \mathbb{N}$. Since we have also assumed that the random parameters entering H are i.i.d., this yields by translation invariance that all H_{Λ_k} , $k = 1, \dots, m$, are simple, again almost surely. Lemma 5.4 when applied to the restrictions H_{Λ_k} of the XY chain H , this gives

$$\Gamma_{\rho_{\alpha(k)}}^{\mathcal{C}^{(k)}} = \chi_{\Delta_{\alpha(k)}}(M_k), \quad k = 1, \dots, m. \quad (5.50)$$

Here M_k is the effective Hamiltonian for H_{Λ_k} , i.e. the restriction of the block-Jacobi matrix (5.5) to Λ_k , $\Delta_{\alpha(k)}$ is a suitable subset of the spectrum of M_k (defined in terms

of the eigenvalues of M_k in analogy to (5.47)), and the right hand side of (5.50) denotes the spectral projection for M_k onto $\Delta_{\alpha^{(k)}}$.

In summary, using (5.45) and (5.50), the correlation matrix of ρ_t is given by

$$\Gamma_{\rho_t}^{\mathcal{C}} = e^{-2itM} \chi_{\alpha} e^{2itM}, \text{ where } \chi_{\alpha} := \bigoplus_{k=1}^m \chi_{\Delta_{\alpha^{(k)}}}(M_k). \quad (5.51)$$

We now prove our main result.

PROOF OF THEOREM 5.1. An important first step is to recall that the eigenvectors ψ_k of the restricted XY chains H_{Λ_k} as eigenvectors of equivalent free Fermion systems (5.36). And each of the eigenvectors ψ_k of H_{Λ_k} in the definition (5.23) of ρ must, up to a phase, be equal to one of the Fermion eigenvectors $\psi_{\alpha^{(k)}}$ and the corresponding eigenprojector equal to $\rho_{\alpha^{(k)}} = |\psi_{\alpha^{(k)}}\rangle\langle\psi_{\alpha^{(k)}}|$. This shows that

$$\rho = \rho_{\alpha} := |\psi_{\alpha}\rangle\langle\psi_{\alpha}|, \quad (5.52)$$

where $\psi_{\alpha} := \psi_{\alpha^{(1)}} \otimes \dots \otimes \psi_{\alpha^{(m)}}$, $\alpha := (\alpha^{(1)}, \dots, \alpha^{(m)}) \in \{0, 1\}^n$. Thus the claim (5.28) is equivalent to

$$\mathbb{E} \left(\sup_{t, \alpha} \mathcal{E}((\rho_{\alpha})_t) \right) \leq C < \infty \quad (5.53)$$

uniformly in n , a and b .

In the following we fix t and α and abbreviate

$$\Gamma := \Gamma_{(\rho_{\alpha})_t}^{\mathcal{C}} = e^{-2itM} \chi_{\alpha} e^{2itM}, \text{ where } \chi_{\alpha} := \bigoplus_{k=1}^m \chi_{\Delta_{\alpha^{(k)}}}(M_k). \quad (5.54)$$

Also, let Γ_1 be the restriction of Γ to $\text{span}\{e_{2j-1}, e_{2j} : j \in \Lambda_0\}$. As explained at the beginning of the section, ρ_t is quasi-free. Thus, by Lemma 5.1, Γ_1 is the $2|\Lambda_0| \times 2|\Lambda_0|$ correlation matrix of the reduced state $(\rho_{\alpha})_t^1$ and

$$\mathcal{E}((\rho_{\alpha})_t) = -\text{tr} [\Gamma_1 \log \Gamma_1]. \quad (5.55)$$

Let the eigenvalues of Γ_1 be $\xi_j, 1 - \xi_j, j = 1, \dots, |\Lambda_0|$, with $0 \leq \xi_j \leq 1/2$.

The following is a calculation essentially taken from [PS14], extended to the more general type of correlation matrices needed here.

We have

$$\begin{aligned}
\mathcal{E}((\rho_\alpha)_t) &= -\operatorname{tr} [\Gamma_1 \log \Gamma_1] \\
&= -\sum_{j=1}^{|\Lambda_0|} (\xi_j \log \xi_j + (1 - \xi_j) \log(1 - \xi_j)) \\
&\leq 2 \log 2 \sum_{j=1}^{|\Lambda_0|} \sqrt{\xi_j(1 - \xi_j)} \\
&= \log 2 \operatorname{tr} [(\Gamma_1(\mathbb{1} - \Gamma_1))^{1/2}],
\end{aligned} \tag{5.56}$$

where we have used the elementary inequality

$$-x \log x - (1 - x) \log(1 - x) \leq 2 \log 2 \sqrt{x(1 - x)}, \quad \text{for } 0 < x < 1. \tag{5.57}$$

The Peierls-Bogoliubov inequality, Appendix B.1 and [Sim], says that

$$\operatorname{tr}[f(A)] \geq \sum_{j=1}^m f(A_{jj}) \tag{5.58}$$

for any convex function f and $m \times m$ hermitian matrix A . Using this with $f(x) = -\sqrt{x}$ as well as the elementary inequality $\sqrt{x} + \sqrt{y} \leq \sqrt{2}\sqrt{x+y}$, we may further bound (5.56) by

$$\mathcal{E}((\rho_\alpha)_t) \leq \sqrt{2} \log 2 \sum_{j \in \Lambda_0} (\operatorname{tr} [(\Gamma_1(\mathbb{1} - \Gamma_1))_{jj}])^{1/2}, \tag{5.59}$$

where matrix elements should be understood as 2×2 -matrices.

Now, since Γ is an orthogonal projection, we use $\Gamma^2 = \Gamma$ with block matrix multiplication to get,

$$\Gamma_{jj} = \Gamma_{jj}^2 + \sum_{k \in \Lambda_0, j \neq k} \Gamma_{jk} \Gamma_{kj} + \sum_{k \in \Lambda \setminus \Lambda_0} \Gamma_{jk} \Gamma_{kj}. \tag{5.60}$$

Thus

$$\Gamma_{jj}(\mathbb{1} - \Gamma_{jj}) = \sum_{k \in \Lambda_0, j \neq k} \Gamma_{jk} \Gamma_{kj} + \sum_{k \in \Lambda \setminus \Lambda_0} \Gamma_{jk} \Gamma_{kj}. \quad (5.61)$$

Then, for $j \in \Lambda_0$,

$$(\Gamma_1(\mathbb{1} - \Gamma_1))_{jj} = \Gamma_{jj}(\mathbb{1} - \Gamma_{jj}) - \sum_{k \in \Lambda_0, k \neq j} \Gamma_{jk} \Gamma_{kj} = \sum_{k \in \Lambda \setminus \Lambda_0} \Gamma_{jk} \Gamma_{kj} = \sum_{k \in \Lambda \setminus \Lambda_0} \Gamma_{jk} (\Gamma_{jk})^*. \quad (5.62)$$

Inserting this into (5.59) and using $\text{tr} [\Gamma_{jk} (\Gamma_{jk})^*] \leq 2 \|\Gamma_{jk}\|^2$, we get

$$\begin{aligned} \mathcal{E}((\rho_\alpha)_t) &\leq \sqrt{2} \log 2 \sum_{j \in \Lambda_0} \left(\sum_{k \in \Lambda \setminus \Lambda_0} \text{tr} [\Gamma_{jk} (\Gamma_{jk})^*] \right)^{\frac{1}{2}} \\ &\leq 2 \log 2 \sum_{j \in \Lambda_0} \left(\sum_{k \in \Lambda \setminus \Lambda_0} \|\Gamma_{jk}\|^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (5.63)$$

And using $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, we find

$$\mathcal{E}((\rho_\alpha)_t) \leq 2 \log 2 \sum_{\ell \in \Lambda_0} \sum_{\ell' \in \Lambda \setminus \Lambda_0} \|\Gamma_{\ell, \ell'}\| \quad (5.64)$$

for all α and t .

Thus we are led to having to bound the expectations of $\|\Gamma_{\ell, \ell'}\|$. Using (5.54), it is clear that

$$\Gamma_{\ell, \ell'} = \sum_{z, z' \in \Lambda} (e^{-2itM})_{\ell, z} (\chi_\alpha)_{z, z'} (e^{2itM})_{z', \ell'}. \quad (5.65)$$

Let us denote by “sup” the supremum over all $\alpha \in \{0, 1\}^n$ and $t \in \mathbb{R}$. The Hölder inequality implies

$$\begin{aligned} \mathbb{E}(\sup \|\Gamma_{\ell, \ell'}\|) &\leq \sum_{z, z' \in \Lambda} \mathbb{E}(\sup \|(e^{-2itM})_{\ell, z}\|^3)^{\frac{1}{3}} \mathbb{E}(\sup \|(\chi_\alpha)_{z, z'}\|^3)^{\frac{1}{3}} \times \\ &\quad \times \mathbb{E}(\sup \|(e^{2itM})_{z', \ell'}\|^3)^{\frac{1}{3}} \end{aligned} \quad (5.66)$$

Since all the operators on the right above have norm bounded by 1, we may estimate $\|\cdot\|^3 \leq \|\cdot\|$ and then apply our decay assumption (5.2) on eigencorrelator localization.

Note that by assuming the validity of (5.2) for *all* n and by the translation invariance of the distribution of random parameters, (5.2) also applies to the effective Hamiltonians M_k of the subsystems. Note that, if $z \in \Lambda_{k_1}$ and $z' \in \Lambda_{k_2}$ such that $k_1 \neq k_2$ then

$$(\chi_\alpha)_{z,z'} = 0.$$

For any $z, z' \in \Lambda_k$, let \tilde{z} and \tilde{z}' be their shift to the set $\{1, 2, \dots, |\Lambda_k|\}$ then

$$\begin{aligned} \mathbb{E}(\sup \|(\chi_\alpha)_{z,z'}\|) &\leq \mathbb{E}\left(\sup \|(\chi_{\Delta_{\alpha(k)}}(M_k))_{\tilde{z},\tilde{z}'}\|\right) \\ &\leq \mathbb{E}\left(\sup_{|g|\leq 1} \|[g(M)]_{\tilde{z},\tilde{z}'}\|\right) \\ &\leq \frac{C}{(1 + |\tilde{z} - \tilde{z}'|)^\beta} = \frac{C}{(1 + |z - z'|)^\beta}. \end{aligned}$$

We also have

$$\mathbb{E}(\sup \|(e^{\pm 2itM})_{j,k}\|) \leq \mathbb{E}\left(\sup_{|g|\leq 1} \|[g(M)]_{j,k}\|\right) \leq \frac{C}{(1 + |j - k|)^\beta}. \quad (5.67)$$

This leads to bounds for all the matrix elements appearing in (5.66).

$$\begin{aligned} \mathbb{E}(\sup \|\Gamma_{\ell,\ell'}\|) &\leq C \sum_{z,z'\in\Lambda} \frac{1}{(1 + |\ell - z|)^{\beta/3}} \frac{1}{(1 + |z - z'|)^{\beta/3}} \frac{1}{(1 + |z' - \ell'|)^{\beta/3}} \\ &\leq C \cdot D \sum_{z\in\mathbb{Z}} \frac{1}{(1 + |\ell - z|)^{\beta/3}} \frac{1}{(1 + |z - \ell'|)^{\beta/3}} \\ &\leq C \cdot D^2 \cdot \frac{1}{(1 + |\ell - \ell'|)^{\beta/3}}. \end{aligned} \quad (5.68)$$

Note that we used the following fact. For any $\beta > 1$, there is a positive number $D < \infty$ for which

$$\sum_{z\in\mathbb{Z}} \frac{1}{(1 + |x - z|)^\beta} \frac{1}{(1 + |z - y|)^\beta} \leq D \frac{1}{(1 + |x - y|)^\beta} \quad \text{for all } x, y \in \mathbb{Z}. \quad (5.69)$$

For example, one may take $D = 2^{\beta+1} \sum_z (1 + |z|)^{-\beta}$, see Lemma B.5.

Inserting the result into (5.64) gives

$$\begin{aligned}\mathbb{E}(\sup \mathcal{E}((\rho_\alpha)_t)) &\leq 2 \log 2 \cdot C \cdot D^2 \sum_{\ell \in \Lambda_0} \sum_{\ell' \in \Lambda \setminus \Lambda_0} \frac{1}{(1 + |\ell - \ell'|)^{\beta/3}} \\ &\leq 4 \log 2 \cdot C \cdot D^2 \sum_{j=1}^{\infty} \frac{j}{(1 + j)^{\beta/3}} =: C' < \infty \text{ if } \beta > 6.\end{aligned}$$

This completes the proof of (5.53). \square

We finally prove Corollaries 5.1 and 5.2

PROOF OF COROLLARY 5.1. In the case of initial condition $\rho = |e_\alpha\rangle\langle e_\alpha|$ given by up-down spins, Γ in (5.54) will be given as

$$\Gamma = e^{-2itM} N_\alpha e^{2itM}, \quad (5.70)$$

where N_α is a form of the number operator,

$$N_\alpha = \text{diag}\{N_{\alpha,j} : j = 1, \dots, n\}, \quad N_{\alpha,j} = \begin{pmatrix} \delta_{\alpha_j,0} & 0 \\ 0 & \delta_{\alpha_j,1} \end{pmatrix}. \quad (5.71)$$

Then (5.65) will read

$$\Gamma_{jk} = \sum_r (e^{-itM})_{jr} N_{\alpha,r} (e^{itM})_{rk}. \quad (5.72)$$

As $\|N_{\alpha,r}\| = 1$ for all r , this leads to an analogue of (5.66),

$$\begin{aligned}\mathbb{E} \left(\sup_{t,\alpha} \|\Gamma_{jk}\| \right) &\leq \sum_r \mathbb{E} \left(\sup_t \|(e^{-itM})_{jr}\| \|(e^{itM})_{rk}\| \right) \\ &\leq \sum_r \left(\mathbb{E}(\sup_t \|(e^{-itM})_{jr}\|^2) \right)^{1/2} \left(\mathbb{E}(\sup_t \|(e^{itM})_{rk}\|^2) \right)^{1/2} \\ &\leq \sum_r \left(\mathbb{E}(\sup_t \|(e^{-itM})_{jr}\|) \right)^{1/2} \left(\mathbb{E}(\sup_t \|(e^{itM})_{rk}\|) \right)^{1/2} \\ &\leq C^2 \sum_r \frac{1}{(1 + |j - r|)^{\beta/2}} \frac{1}{(1 + |r - k|)^{\beta/2}} \\ &\leq C^2 \cdot D \frac{1}{(1 + |j - k|)^{\beta/2}}.\end{aligned} \quad (5.73)$$

This will give an area law requiring $\beta > 4$:

$$\begin{aligned} \mathbb{E}(\sup \mathcal{E}((\rho_\alpha)_t)) &\leq 2 \log 2 \cdot C^2 \cdot D \sum_{j \in \Lambda_0, k \in \Lambda \setminus \Lambda_0} \frac{1}{(1 + |j - k|)^{\beta/2}} \\ &\leq 4 \log 2 \cdot C^2 \cdot D \sum_{j=1}^{\infty} \frac{j}{(1 + j)^{\beta/2}}. \end{aligned} \quad (5.74)$$

□

Finally, we prove Corollary 5.2.

PROOF OF COROLLARY 5.2. We simply get

$$\Gamma = \chi_{\Delta_\alpha}(M). \quad (5.75)$$

This gives that

$$\mathbb{E} \left(\max_\alpha \mathcal{E}(\rho_\alpha) \right) \leq 2 \log 2 \sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} \mathbb{E} \left(\sup_\alpha \left\| [\chi_{\Delta_\alpha}(M)]_{jk} \right\| \right). \quad (5.76)$$

Now, by assumption (5.2),

$$\mathbb{E} \left(\max_\alpha \left\| [\chi_{\Delta_\alpha}(M)]_{jk} \right\| \right) \leq \mathbb{E} \left(\sup_{|g| \leq 1} \left\| [g(M)]_{jk} \right\| \right) \leq \frac{C}{(1 + |j - k|)^\beta} \quad (5.77)$$

for some $\beta > 2$. We have

$$\begin{aligned} \sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} \frac{1}{(1 + |j - k|)^\beta} &\leq \sum_{j=1}^{\ell} \sum_{k \in \mathbb{Z} \setminus \{1, \dots, \ell\}} \frac{1}{(1 + |j - k|)^\beta} \\ &= 2 \sum_{j=1}^{\ell} \sum_{k=\ell+1}^{\infty} \frac{1}{(1 + (k - j))^\beta} \\ &\leq 2 \sum_{j=1}^{\ell} \sum_{k=\ell+1}^{\infty} \frac{1}{(1 + (\ell - j))^{\frac{\beta}{2}}} \frac{1}{(1 + (k - \ell))^{\frac{\beta}{2}}} \\ &\leq 2 \left(\sum_{j=0}^{\infty} \frac{1}{(1 + j)^{\frac{\beta}{2}}} \right)^2 < \infty. \end{aligned} \quad (5.78)$$

In the second to third step we used the following inequality: for $j \in \{1, \dots, \ell\}$ and $k \geq \ell + 1$,

$$\begin{aligned} \frac{1}{(1 + (k - j))^\beta} &= \frac{1}{(1 + (k - j))^{\frac{\beta}{2}}} \frac{1}{(1 + (k - j))^{\frac{\beta}{2}}} \\ &\leq \frac{1}{(1 + (\ell - j))^{\frac{\beta}{2}}} \frac{1}{(1 + (k - \ell))^{\frac{\beta}{2}}}. \end{aligned}$$

This gives the uniform boundedness of (5.76) in n , r and ℓ , and thus completes the proof. \square

Next we include a proof of Theorem 5.2 which is a variation of the proof of Theorem 5.1. Again, we include only these variations.

PROOF OF THEOREM 5.2. As recalling from (5.64), and raising both sides to power p ,

$$\begin{aligned} (\sup \mathcal{E}((\rho_\alpha)_t))^p &\leq \left(2 \log 2 \sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} \sup \|\Gamma_{j,k}\| \right)^p \\ &= (2 \log 2)^p \sum_{j_1, j_2, \dots, j_p \in \Lambda_0} \sum_{k_1, \dots, k_p \in \Lambda \setminus \Lambda_0} \prod_{\ell=1}^p \sup \|\Gamma_{j_\ell, k_\ell}\|. \end{aligned} \quad (5.79)$$

As before, we used “sup” to denote the supremum over $\alpha \in \{0, 1\}^n$ and $t \in \mathbb{R}$. Then by averaging, we find

$$\begin{aligned} \mathbb{E}((\sup \mathcal{E}((\rho_\alpha)_t))^p) &\leq (2 \log 2)^p \sum_{j_1, j_2, \dots, j_p \in \Lambda_0} \sum_{k_1, \dots, k_p \in \Lambda \setminus \Lambda_0} \mathbb{E} \left(\prod_{\ell=1}^p \sup \|\Gamma_{j_\ell, k_\ell}\| \right) \\ &\leq (2 \log 2)^p \sum_{j_1, j_2, \dots, j_p \in \Lambda_0} \sum_{k_1, \dots, k_p \in \Lambda \setminus \Lambda_0} \prod_{\ell=1}^p [\mathbb{E}((\sup \|\Gamma_{j_\ell, k_\ell}\|)^p)]^{\frac{1}{p}} \\ &= (2 \log 2)^p \left(\sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} [\mathbb{E}((\sup \|\Gamma_{j,k}\|)^p)]^{\frac{1}{p}} \right)^p \\ &\leq (2 \log 2)^p \left(\sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} [\mathbb{E}(\sup \|\Gamma_{j,k}\|)]^{\frac{1}{p}} \right)^p \end{aligned} \quad (5.80)$$

where we used Hölder in the second step, in the last step we used that $\sup \|\Gamma_{j,k}\| \leq 1$.

Recall from (5.68) that

$$\mathbb{E}(\sup \|\Gamma_{j,k}\|) \leq \frac{C \cdot D^2}{(1 + |j - k|)^{\frac{\beta}{3}}}.$$

Thus,

$$[\mathbb{E}((\sup \mathcal{E}((\rho_\alpha)_t)^p)]^{\frac{1}{p}}] \leq 2 \log 2 \cdot (C \cdot D^2)^{1/p} \sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} \frac{1}{(1 + |j - k|)^{\frac{\beta}{3p}}}. \quad (5.81)$$

Following the same steps as before, the right hand side of (5.81) is finite if $\beta > 6p$ as claimed. \square

Finally, we include the proof of Corollary 5.3.

PROOF OF COROLLARY 5.3. First we find

$$\begin{aligned} \mathbb{E}(\sup \|\Gamma_{j,k}\|) &\leq C \sum_{z, z' \in \Lambda} e^{-\frac{\eta}{3}|j-z|} e^{-\frac{\eta}{3}|z-z'|} e^{-\frac{\eta}{3}|z'-k|} \\ &\leq C \cdot \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{3}|z|} \right) \sum_z e^{-\frac{\eta}{3}|j-z|} e^{-\frac{\eta}{6}|z-k|} \\ &\leq C \cdot \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{3}|z|} \right) \sum_z e^{-\frac{\eta}{6}|j-z|} e^{-\frac{\eta}{6}|z-k|} \\ &\leq C \cdot \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{3}|z|} \right) \left(\sum_{z \in \mathbb{Z}} e^{-\frac{\eta}{6}|z|} \right) e^{-\frac{\eta}{12}|j-k|} \\ &=: C \cdot D' e^{-\frac{\eta}{12}|j-k|} \end{aligned} \quad (5.82)$$

where we used the following inequality: for any $\zeta > 0$ and $x, y \in \mathbb{Z}$,

$$\begin{aligned} \sum_{z \in \mathbb{Z}} e^{-\zeta|x-z|} e^{-\zeta|z-y|} &\leq e^{-\frac{\zeta}{2}|x-y|} \sum_{z \in \mathbb{Z}} e^{-\frac{\zeta}{2}|x-z|} e^{-\frac{\zeta}{2}|z-y|} \\ &\leq e^{-\frac{\zeta}{2}|x-y|} \left(\sum_z e^{-\zeta|x-z|} \right)^{\frac{1}{2}} \left(\sum_z e^{-\zeta|z-y|} \right)^{\frac{1}{2}} \\ &= \left(\sum_{z \in \mathbb{Z}} e^{-\zeta|z|} \right) e^{-\frac{\zeta}{2}|x-y|}. \end{aligned} \quad (5.83)$$

Inserting this in (5.80), we find

$$\mathbb{E}(\sup \mathcal{E}((\rho_\alpha)_t)^p) \leq (2 \log 2)^p \cdot C \cdot D' \left(\sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} e^{-\frac{\eta}{12p}|j-k|} \right)^p. \quad (5.84)$$

Then by taking the p -th root

$$\begin{aligned} [\mathbb{E}(\sup \mathcal{E}((\rho_\alpha)_t)^p)]^{\frac{1}{p}} &\leq 2 \log 2 \cdot (C \cdot D')^{1/p} \sum_{j \in \Lambda_0} \sum_{k \in \Lambda \setminus \Lambda_0} e^{-\frac{\eta}{12p}|j-k|} \\ &\leq 4 \log 2 \cdot (C \cdot D')^{1/p} \sum_{j=0}^{\ell} \sum_{k=\ell+1}^{\infty} e^{-\frac{\eta}{12p}(k-j)} \\ &= 4 \log 2 \cdot (C \cdot D')^{1/p} \sum_{j=0}^{\ell} \sum_{k=\ell+1}^{\infty} e^{-\frac{\eta}{24p}(k-\ell)} e^{-\frac{\eta}{24p}(\ell-j)} \\ &\leq 4 \log 2 \cdot (C \cdot D')^{1/p} \sum_{k=\ell+1}^{\infty} e^{-\frac{\eta}{24p}(k-\ell)} \sum_{j=0}^{\ell} e^{-\frac{\eta}{24p}(\ell-j)} \\ &\leq 4 \log 2 \cdot (C \cdot D')^{1/p} \left(\sum_{j=0}^{\infty} e^{-\frac{\eta}{24p}j} \right)^2 \\ &= 4 \log 2 \cdot (C \cdot D')^{1/p} \left(\frac{1}{1 - e^{-\frac{\eta}{24p}}} \right)^2. \end{aligned}$$

And this completes the proof. □

CHAPTER 6

A Brief Discussion of Current and Future Work

We proved area laws for dynamical entanglement for a class of initial product pure states. In particular, these pure states are the eigenstates of restrictions of the disordered XY chain to a block of spins. A key tool was the theory of Fermionic operators with quasi-free states.

A natural question to ask: What if the initial state is a tensor product of generic pure states, or for what other initial pure states one can prove a bounded dynamical entanglement? One may also wonder if more general initial pure states could provide examples where the dynamical entanglement is not uniformly bounded in time, for example in the form of logarithmic growth, as the physicists believe to happen in some models. But it's not clear if this happens in XY or if one has to consider more complicated models. Our methods cannot be applied for states that are not quasi-free states where the crucial formula in Theorem 3.8 breaks down.

However, thermal states are quasi-free states. Thus one hopes to go beyond initial pure states and ask about the dynamical entanglement of an initial product of thermal states. But this leads us to the conflict that the entanglement entropy is not a good tool to quantify entanglements of mixed states. The best available tool is the so-called *logarithmic negativity* defined as the logarithm of the trace norm of the partial transpose of the state with respect to one of the two subsystems, see for example [Ple05]. The problem here is that finding the logarithmic negativity for free Fermion systems is an open problem, recent advances [EZ15, CW16] did not resolve the problem completely but can serve as a good starting point. Thus, we have troubles with a more elementary situation, where proving area law for the entanglement of thermal states of the whole system is still not accessible. This is an

existing interesting open conjecture since [SW16b] proved the exponential decay of correlations of thermal states of free Fermion systems.

So far, our results and recent results left very few questions to ask about XY chains from the MBL perspective. We need to find other many-body models where rigorous results can be proven. Currently, those are the systems that reduce to an effective single particle Hamiltonian, otherwise the results are on the level of computations and physical heuristics. We are working on the disordered Harmonic oscillator systems, where area laws have been proven for the ground and the thermal states, see [NSS12, NSS13]. Our goal is to say something about the entanglement of the excited states that correspond to an interval at the bottom of the spectrum. More ambitious plans include the challenging problem of developing the mathematical machinery needed to prove any of the MBL manifestations for the physically interesting XXZ model.

LIST OF REFERENCES

- [ARH16] D. Abanin, W. De Roeck, and F. Huveneers. A theory of many-body localization in periodically driven systems. *Annals of Physics (to appear)*, 2016.
- [ARNSS16] H. Abdul-Rahman, B. Nachtergaele, R. Sims, and G. Stolz. Entanglement dynamics of disordered quantum XY chains. *Letter in Mathematical Physics*, 106:649–674, 2016.
- [ARRS99] T. Antal, Z. Rácz, A. Rákos, and G. M. Schütz. Transport in the XX chain at zero temperature: Emergence of flat magnetization profiles. *Physical Review E*, 59:4912–4918, 1999.
- [ARRS08] T. Antal, Z. Rácz, A. Rákos, and G. M. Schütz. Isotropic transverse XY chain with energy and magnetization currents. *Physical Review E*, 59:5184–5189, 2008.
- [ARS15] H. Abdul-Rahman and G. Stolz. A uniform area law for the entanglement of eigenstates in the disordered XY chain. *Journal of Mathematical Physics*, 56:121901, 2015.
- [Att] S. Attal. *Lectures in quantum noise theory*. <http://math.univ-lyon1.fr/~attal/chapters.html>.
- [AW09] M. Aizenman and S. Warzel. Localization bounds for multiparticle systems. *Communications in Mathematical Physics*, 290:903–934, 2009.
- [BAA06] D. M. Basko, I. L. Aleiner, and B. L. Altshuler. Metal-insulator transition in a weakly interacting many-electron system with localized single-particle states. *Annals of Physics*, 321:1126–1205, 2006.
- [BH13] F. Brandao and M. Horodecki. An area law for entanglement from exponential decay of correlations. *Nature Physics*, 9:721–726, 2013.
- [BH15] F. Brandao and M. Horodecki. Exponential decay of correlations implies area law. *Communications in Mathematical Physics*, 333:761–798, 2015.
- [BLS94] V. Bach, E. H. Lieb, and J. P. Solovej. Generalized Hartree-Fock theory and the Hubbard model. *Journal of Statistical Physics*, 76:3–89, 1994.
- [BN13] B. Bauer and C. Nayak. Area laws in a many-body localized state and its implications for topological order. *Journal of Statistical Mechanics*, 2013. P09005.

- [BO07] C. Burrell and T. Osborne. Bounds on the speed of information propagation in disordered quantum spin chains. *Physical Review Letters*, 99:167201, 2007.
- [BPM12] J. H. Bardarson, F. Pollmann, and J. E. Moore. Unbounded growth of entanglement in models of many-body localization. *Physical Review Letters*, 109:017202, 2012.
- [BR97] O. Bratteli and D. Robinson. *Operator algebras and quantum statistical mechanics*, volume 2. Springer Verlag, New York, NY, 2 edition, 1997.
- [CMCF06] G. De Chiara, S. Montanegro, P. Calabrese, and R. Fazio. Entanglement entropy dynamics of Heisenberg chains. *Journal of Statistical Mechanics*, (P03001), 2006.
- [CRF⁺11] E. Canovi, D. Rossini, R. Fazio, G. E. Santoro, and A. Silva. Quantum quenches, thermalization and many-body localization. *Physical Review B*, 83:094431, 2011.
- [CS09a] V. Chulaevsky and Y. Suhov. Eigenfunctions in a two-particle Anderson tight binding model. *Communications in Mathematical Physics*, 289:701–723, 2009.
- [CS09b] V. Chulaevsky and Y. Suhov. Multi-particle Anderson localisation: induction on the number of particles. *Mathematical Physics, Analysis and Geometry*, 12:117–139, 2009.
- [CS15] J. Chapman and G. Stolz. Localization for random block operators related to the XY spin chain. *Annals Henri Poincaré*, 16:405–435, 2015.
- [CTDL91] C. Cohen-Tannoudji, B. Diu, and F. Laloe. *Quantum mechanics*, volume 2. Wiley, Paris, 1991.
- [CW16] P. Chang and X. Wen. Entanglement negativity in free-fermion systems: an overlap matrix approach. *arXiv:1601.07492*, 2016.
- [EPS16] A. Elgart, L. Pastur, and M. Shcherbina. Large block properties of the entanglement entropy of disordered fermions. *arXiv:1601.00294*, 2016.
- [ESS14] A. Elgart, M. Shamis, and S. Sodin. Localisation for non-monotone Schrödinger operators. *The Journal of the European Mathematical Society*, 16:909924, 2014.
- [EZ15] V. Eisler and Z. Zimboras. On the partial transpose of fermionic gaussian states. *New Journal of Physics*, 17:053048, 2015.
- [FWB⁺15] M. Friesdorf, A. H. Werner, W. Brown, V. B. Scholz, and J. Eisert. Many-body localisation implies that eigenvectors are matrix-product states. *Physical Review Letters*, 114:170505, 2015.
- [GE16] C. Gogolin and J. Eisert. Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems. *Reports on Progress in Physics*, 79:056001, 2016.
- [GL16] M. Gebert and M. Lemm. On polynomial Lieb-Robinson bounds for the XY chain in a decaying random field. *arXiv:1601.03383*, 2016.

- [GMP05] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov. Interacting electrons in disordered wires: Anderson localization and low- t transport. *Physical Review Letters*, 95:206603, 2005.
- [Has07] M. Hastings. An area law for one dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiments*, 2007. P08024.
- [HNO14] D. A. Huse, R. Nandkishore, and V. Oganesyan. Phenomenology of fully many-body-localized systems. *Physical Review B*, 90:174202, 2014.
- [HSS12] E. Hamza, R. Sims, and G. Stolz. Dynamical localization in disordered quantum spin systems. *Communications in Mathematical Physics*, 315:215–239, 2012.
- [IJK05] A. R. Its, B.-Q. Jin, and V. E. Korepin. Entanglement in XY spin chain. *Journal of Physics A: Mathematical and General*, 38:2975–2990, 2005.
- [Imb16] J. Z. Imbrie. On many-body localization for quantum spin chains. *Journal of Statistical Physics*, 163:998–1048, 2016.
- [IMM08] A. R. Its, F. Mezzadri, and M. Y. Mo. Entanglement entropy in quantum spin chains with finite range interaction. *Communications in Mathematical Physics*, 284:117185, 2008.
- [JW28] P. Jordan and E. Wigner. Über das paulische Äquivalenzverbot. *Zeitschrift für Physik*, 47:631–651, 1928.
- [KN13] A. Klein and S. T. Nguyen. The bootstrap multiscale analysis for the multi-particle Anderson model. *Journal of Statistical Physics*, 151:938–973, 2013.
- [KP90] A. Klein and J. F. Perez. Localization in the ground-state of the one dimensional XY model with a random transverse field. *Communications in Mathematical Physics*, 128:99–108, 1990.
- [LSM61] E. Lieb, T. Schultz, and D. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16:407–466, 1961.
- [Mat05] Sho Matsumoto. α -Pfaffian, Pfaffian point process and shifted Schur measure. *Linear Algebra and its Applications*, 403:369–398, 2005.
- [MM60] W. H. Metzler and T. Muir. *A treatise on the theory of determinants*. Dover Publications, New York, 1960.
- [MSA13] Z. Papić, M. Serbyn, and D. A. Abanin. Local conservation laws and the structure of the many-body localized states. *Physical Review Letters*, 111:127201, 2013.
- [NH15] R. Nandkishore and D. Huse. Many-body localization and thermalization in quantum statistical mechanics. *Annual Review of Condensed Matter Physics*, 6:15–38, 2015.

- [Nie05] M. A. Nielsen. *The Fermionic canonical commutation relations and the Jordan-Wigner transform*. 2005.
- [NSS12] B. Nachtergaele, R. Sims, and G. Stolz. Quantum harmonic oscillator systems with disorder. *Journal of Statistical Physics*, 149:969–1012, 2012.
- [NSS13] B. Nachtergaele, R. Sims, and G. Stolz. An area law for the bipartite entanglements of disordered oscillator systems. *Journal of Mathematical Physics*, 54:042110, 2013.
- [Oga02] Y. Ogata. Diffusion of the magnetization profile in the XX model. *Physical Review E*, page 066123, 2002.
- [OH07] V. Oganesyan and D. A. Huse. Localization of interacting fermions at high temperature. *Physical Review B*, 75:155111, 2007.
- [Pet08] D. Petz. *Quantum information theory and quantum statistics*. Springer, Berlin Heidelberg, 2008.
- [PH10] A. Pal and D. A. Huse. The many-body localization phase transition. *Physical Review B*, 82:174411, 2010.
- [Ple05] M. Plenio. The logarithmic negativity: A full entanglement monotone that is not convex. *Physical Review Letters*, 95:090503, 2005.
- [PS14] L. Pastur and V. Slavin. On the area law for disordered free fermions. *Physical Review Letters*, 113:150404, 2014.
- [RH15] W. De Roeck and F. Huveneers. Can translation invariant systems exhibit a many-body localized phase? *From Particle Systems to Partial Differential Equations II, Springer Proceedings in Mathematics & Statistics*, 129:173–192, 2015.
- [RHMS16] W. De Roeck, F. Huveneers, M. Müller, and M. Schiulaz. Absence of many-body mobility edges. *Physical Review B*, 93:014203, 2016.
- [RS75] M. Reed and B. Simon. *Methods of modern mathematical physics I: functional analysis*. Academic Press Inc., New York, 1975.
- [Sim] B. Simon. *Trace ideals and their applications*. American Mathematical Society, 2 edition.
- [Sim93] B. Simon. *The statistical mechanics of lattice gases*, volume 1. Princeton University Press, Princeton, 1993.
- [SKG⁺14] M. Serbyn, M. Knap, S. Gopalakrishnan, Z. Papić, N. Y. Yao, C. R. Laumann, D. A. Abanin, M. D. Lukin, and E. A. Demler. Interferometric probes of many-body localization. *Physical Review Letters*, 113:147204, 2014.
- [SN11] J. J. Sakurai and J. Napolitano. *Modern quantum mechanics*. Addison-Wesley, 2 edition, 2011.

- [SPA13] M. Serbyn, Z. Papić, and D. A. Abanin. Universal slow growth of entanglement in interacting strongly disordered systems. *Physical Review Letters*, 110:260601, 2013.
- [Ste90] J. R. Stembridge. Nonintersecting paths, pfaffians, and plane partitions. *Advances in Mathematics*, 83:96–131, 1990.
- [Sto11] G. Stolz. An introduction to the mathematics of Anderson localization. *Entropy and the Quantum II. Contemporary Mathematics: American Mathematical Society*, 552:71–108, 2011.
- [SW16a] R. Seiringer and S. Warzel. Decay of correlations and absence of superfluidity in the disordered Tonks-Girardeau gas. *New Journal of Physics*, 18:035002, 2016.
- [SW16b] R. Sims and S. Warzel. Decay of determinantal and pfaffian correlation functionals in one-dimensional lattices. *Communications in Mathematical Physics (to appear)*, 2016.
- [VA13] R. Vosk and E. Altman. Many-body localization in one dimension as a dynamical renormalization group fixed point. *Physical Review Letters*, 110:067204, 2013.
- [VLRK03] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev. Entanglement in quantum critical phenomena. *Physical Review Letters*, 90:227902, 2003.
- [Wei80] J. Weidmann. *Linear operators in Hilbert spaces*. Springer, 1980.
- [ZPP08] M. Znidarić, T. Prosen, and P. Prelovšek. Many-body localization in the Heisenberg XXZ magnet in a random field. *Physical Review B*, 77:064426, 2008.

APPENDIX A

Basics of Pfaffians

In this appendix, we present without proofs some basic facts about Pfaffians of anti-symmetric matrices, more about Pfaffians can be found for example in [MM60, Ste90, Mat05]. Let $A = (a_{ij})$ be an complex $n \times n$ anti-symmetric matrix, i.e. $A^t = -A$ and

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ -a_{12} & 0 & a_{23} & \dots & a_{2n} \\ -a_{13} & -a_{23} & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & -a_{3n} & \dots & 0 \end{bmatrix} \quad (\text{A.1})$$

The Pfaffian of A , denoted by $\text{pf}[A]$ is defined as follows:

If $n \geq 1$ is odd, $\text{pf}[A] = 0$.

If $n = 2m$ for some $m \geq 1$, then

$$\text{pf}[A] = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \text{sgn}(\sigma) \prod_{j=1}^m a_{\sigma(2j-1), \sigma(2j)} \quad (\text{A.2})$$

where S_{2m} is the symmetric group of permutations and $\text{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_{2m}$. Also, the Pfaffian of the 0×0 matrix is defined to be 1. For example, for any $\alpha \in \mathbb{C}$ and

$$A = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}.$$

Then using (A.2), it is clear that

$$\text{pf}[A] = \frac{1}{2}(a_{12} - a_{21}) = \alpha.$$

For 4×4 matrices, the calculations become cumbersome. But we have a useful fact, in analogue to determinants, Pfaffians also satisfy a Laplace expansion, as follows: Let $m \geq 1$ and let A be $2m \times 2m$ anti-symmetric. Then

$$\text{pf}[A] = \sum_{\ell=2}^{2m} (-1)^\ell a_{1,\ell} \text{pf}[A_{\hat{1},\hat{\ell}}] \quad (\text{A.3})$$

where $A_{\hat{1},\hat{\ell}}$ is the matrix obtained from A by simultaneously removing the two rows and the two columns corresponding to 1 and ℓ . This expansion allows for more efficient calculations. For example, let us consider any 4×4 anti-symmetric A

$$A = \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix}$$

Then

$$\begin{aligned} \text{pf}[A] &= a \cdot \text{pf} \left[\begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix} \right] - b \cdot \text{pf} \left[\begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \right] + c \cdot \text{pf} \left[\begin{pmatrix} 0 & d \\ -d & 0 \end{pmatrix} \right] \\ &= af - be + cd. \end{aligned}$$

APPENDIX B

Some Inequalities

In this appendix we prove some inequalities used in the thesis.

B.1. Peierls-Bogoliubov inequality

We start with the Peierls-Bogoliubov Inequality used in the proof of the dynamical entanglement results. This inequality is proven in Section 8.3 from [Sim], and we provide here a different proof.

LEMMA B.1. *Let A be an $n \times n$ Hermitian matrix and f is a convex function then*

$$\mathrm{tr}[f(A)] \geq \sum_{j=1}^n f(A_{jj}). \quad (\text{B.1})$$

PROOF. Since A is Hermitian then there exist a complete set of eigenvectors, i.e. there is an ONB $\{\phi_j : j = 1, 2, \dots, n\}$ such that $A\phi_j = \lambda_j\phi_j$, where $\lambda_j \in \mathbb{R}$. Let $\{e_j : j = 1, 2, \dots, n\}$ be the canonical basis of \mathbb{C}^n , then

$$\begin{aligned} \langle e_j, Ae_j \rangle &= \sum_{k=1}^n \langle e_j, \phi_k \rangle \langle \phi_k, Ae_j \rangle = \sum_{k=1}^n \langle e_j, \phi_k \rangle \langle A\phi_k, e_j \rangle \\ &= \sum_{k=1}^n \lambda_k |\langle e_j, \phi_k \rangle|^2, \end{aligned} \quad (\text{B.2})$$

where we expanded e_j in terms of the ϕ_j 's in the first step. and since $\sum_k |\langle e_j, \phi_k \rangle|^2 = \|\phi\|^2 = 1$ and f is convex, we get

$$\sum_{j=1}^n f(\langle e_j, Ae_j \rangle) \leq \sum_{j=1}^n \sum_{k=1}^n f(\lambda_k) |\langle e_j, \phi_k \rangle|^2 = \sum_{k=1}^n f(\lambda_k) = \mathrm{tr}[f(A)]. \quad (\text{B.3})$$

□

B.2. Other inequalities

The following inequality is used in the proof of Theorem 3.4.

LEMMA B.2. *Let $A \geq 0$, $B \geq 0$, if $A \leq c\mathbb{1}$ where $c \in \mathbb{R}$ then $\text{tr}[AB] \leq c \text{tr}[B]$.*

PROOF. We have

$$A \leq c\mathbb{1} \Rightarrow (c\mathbb{1} - A) \geq 0 \Rightarrow \langle x, (c\mathbb{1} - A)x \rangle \geq 0, \quad \text{for all } x. \quad (\text{B.4})$$

Thus

$$\begin{aligned} \langle B^{\frac{1}{2}}x, (c\mathbb{1} - A)B^{\frac{1}{2}}x \rangle \geq 0 &\Rightarrow \langle x, B^{\frac{1}{2}}(c\mathbb{1} - A)B^{\frac{1}{2}}x \rangle \geq 0 & (\text{B.5}) \\ &\Rightarrow \langle x, cBx \rangle \geq \langle x, B^{\frac{1}{2}}AB^{\frac{1}{2}}x \rangle \\ &\Rightarrow B^{\frac{1}{2}}AB^{\frac{1}{2}} \leq cB \\ &\Rightarrow \text{tr} \left[B^{\frac{1}{2}}AB^{\frac{1}{2}} \right] = \text{tr}[AB] \leq c \text{tr}[B]. \end{aligned}$$

□

The inequalities in Lemmas B.3 and B.4 below are used in the proof of Theorem 4.1, see Section 4.5.

LEMMA B.3. *Let A and B be two $n \times n$ matrices, then*

$$|\text{tr}[AB]| \leq \|A\|_1 \|B\|$$

The proof of Lemma B.3 below extends for general trace class operator A and bounded operator B .

PROOF. Let $s_1 \geq s_2 \geq \dots \geq s_n \geq 0$ be the singular values of A . Then by the SVD of A , there exist ONB $\{g_j\}$ and $\{h_k\}$ of \mathbb{C}^n , such that

$$A = \sum_{j=1}^n s_j |g_j\rangle\langle h_j|.$$

Thus,

$$\begin{aligned}
\operatorname{tr}[AB] &= \sum_{k=1}^n \langle g_k, ABg_k \rangle \\
&= \sum_{k=1}^n \sum_{j=1}^n s_j \langle g_k, g_j \rangle \langle h_j, Bg_k \rangle \\
&= \sum_{j=1}^n s_j \langle h_j, Bg_j \rangle
\end{aligned}$$

This yields,

$$|\operatorname{tr}[AB]| \leq \sum_{j=1}^n s_j \|B\| = \|A\|_1 \|B\|. \quad (\text{B.6})$$

□

LEMMA B.4. *Let A be any $n \times n$ matrix, then*

$$\|A\|_1 \leq \sum_{j,k=1}^n |A_{jk}|. \quad (\text{B.7})$$

The proof of Lemma B.4 below extends for general trace class operators A , and in the case where the matrix elements are considered with respect to any basis.

PROOF. Let $A = USV^*$ be the SVD decomposition of A , where U and V are orthogonal matrices and S is the diagonal matrix of singular values. Thus $S = U^*AV$, then using the ONB $\{U^*e_j\}_{j=1}^n$ where $\{e_j\}$ are the canonical basis of \mathbb{C}^n

$$\begin{aligned}
\|A\|_1 = \operatorname{tr}[S] &= \sum_{j=1}^n \langle U^*e_j, U^*AVU^*e_j \rangle = \sum_{j=1}^n \langle e_j, AVU^*e_j \rangle \quad (\text{B.8}) \\
&\leq \sum_{j=1}^n |\langle e_j, AVU^*e_j \rangle| \\
&\leq \sum_{j=1}^n \sum_{k=1}^n |\langle e_k, VU^*e_j \rangle \langle e_j, Ae_k \rangle| \\
&\leq \sum_{j=1}^n \sum_{k=1}^n |\langle e_j, Ae_k \rangle|.
\end{aligned}$$

In the second-to-third step, we expanded VU^*e_j in terms of the canonical bases $\{e_k\}$. □

The following inequality is used in Section 5.6.

LEMMA B.5. *For any x, y in \mathbb{Z} , and $\beta > 1$ the following inequality is satisfied*

$$\sum_{z \in \mathbb{Z}} \frac{1}{(1 + |x - z|)^\beta} \frac{1}{(1 + |z - y|)^\beta} \leq D \frac{1}{(1 + |x - y|)^\beta} \quad \text{for all } x, y \in \mathbb{Z} \quad (\text{B.9})$$

where $D = 2^{\beta+1} \sum_z (1 + |z|)^{-\beta}$.

PROOF. First, note that proving the inequality is equivalent to prove that for any nonnegative integer x and $\beta > 1$ we have

$$\sum_{z \in \mathbb{Z}} \frac{1}{(1 + |z|)^\beta} \frac{1}{(1 + |z - x|)^\beta} \leq \frac{2^{\beta+1}}{(1 + x)^\beta} \sum_{z \in \mathbb{Z}} \frac{1}{(1 + |z|)^\beta}. \quad (\text{B.10})$$

We will split the left hand sum as follows

$$\sum_{z \in \mathbb{Z}} = \sum_{z > \frac{3x}{2}} + \sum_{z < -\frac{x}{2}} + \sum_{z \in [-\frac{x}{2}, \frac{x}{2}]} + \sum_{z \in (\frac{x}{2}, \frac{3x}{2}]} . \quad (\text{B.11})$$

Of course, the intervals in (B.11) mean their intersection with the set of integers.

For $z > \frac{3x}{2}$, we have $|z| > x$, thus

$$\begin{aligned} \sum_{z > \frac{3x}{2}} \frac{1}{(1 + |z|)^\beta} \frac{1}{(1 + |z - x|)^\beta} &< \sum_{z > \frac{3x}{2}} \frac{1}{(1 + x)^\beta} \frac{1}{(1 + |z - x|)^\beta} \\ &= \frac{1}{(1 + x)^\beta} \sum_{z > \frac{x}{2}} \frac{1}{(1 + |z|)^\beta}. \end{aligned} \quad (\text{B.12})$$

For $z < -\frac{x}{2}$, we have $|z - x| > x$, then a similar argument leads to

$$\sum_{z < -\frac{x}{2}} \frac{1}{(1 + |z|)^\beta} \frac{1}{(1 + |z - x|)^\beta} < \frac{1}{(1 + x)^\beta} \sum_{z < -\frac{x}{2}} \frac{1}{(1 + |z|)^\beta}. \quad (\text{B.13})$$

For $z \in [-\frac{x}{2}, \frac{x}{2}]$, we have $|x - z| \geq \frac{x}{2}$, thus

$$\begin{aligned} \sum_{z \in [-\frac{x}{2}, \frac{x}{2}]} \frac{1}{(1 + |z|)^\beta} \frac{1}{(1 + |z - x|)^\beta} &< \sum_{z \in [-\frac{x}{2}, \frac{x}{2}]} \frac{2^\beta}{(1 + x)^\beta} \frac{1}{(1 + |z|)^\beta} \\ &= \frac{2^\beta}{(1 + x)^\beta} \sum_{z \in [-\frac{x}{2}, \frac{x}{2}]} \frac{1}{(1 + |z|)^\beta}. \end{aligned} \quad (\text{B.14})$$

Similarly, for $z \in (\frac{x}{2}, \frac{3x}{2}]$, $|z| > \frac{x}{2}$ and we get

$$\sum_{z \in (\frac{x}{2}, \frac{3x}{2}]} \frac{1}{(1 + |z|)^\beta} \frac{1}{(1 + |z - x|)^\beta} < \frac{2^\beta}{(1 + x)^\beta} \sum_{z \in (-\frac{x}{2}, \frac{x}{2}]} \frac{1}{(1 + |z|)^\beta}. \quad (\text{B.15})$$

Finally, the substitution of (B.12), (B.13), (B.14), and (B.15) in (B.11) gives our inequality (B.10). \square