

MATH 250b – Section 2

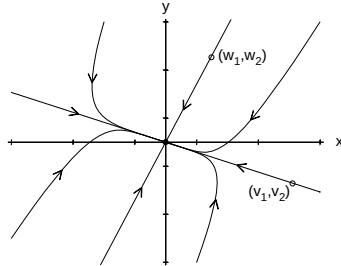
Spring Semester 2008

Tuesday, 29-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

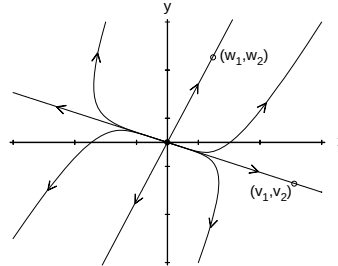
Linear Plane Autonomous Systems $\hat{x}' = A\hat{x}$

***Real
Negative***



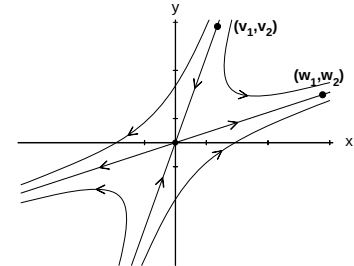
Stable Node

***Real
Positive***



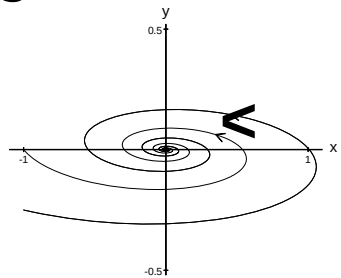
Unstable Node

***Real
Opposite Sign***



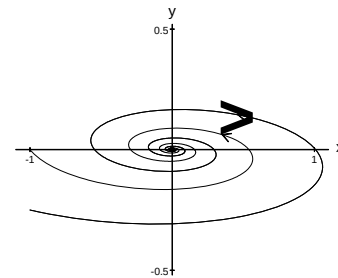
***Saddle
(unstable)***

***Complex
Negative Real Part***



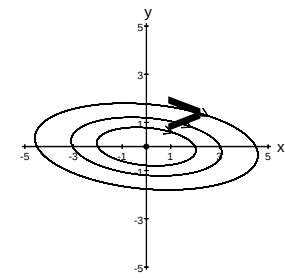
Stable Spiral

***Complex
Positive Real Part***



Unstable Spiral

***Complex
Zero Real Part***



***Center
(neutrally stable)***

Linear Plane Autonomous Systems $\hat{x}' = A\hat{x}$

NOTES

- *Stability* can be determined from the *signs of the real parts* of the eigenvalues alone (provided none has real part 0)
 - stable if all **both** eigenvalues satisfy $\text{Re } \lambda < 0$*
 - unstable if at **least one** eigenvalue satisfies $\text{Re } \lambda > 0$*
- Second order equation phase portraits have the same classifications
(convert to a first order system)

Nonlinear Plane Autonomous Systems

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Equilibrium Equations

$$f(x, y) = 0$$

$$g(x, y) = 0$$

The *Jacobian* $J(x, y)$ evaluated at an equilibrium (x, y) is the *coefficient matrix of the linearization* (at that equilibrium)

$$J(x, y) = \begin{pmatrix} \frac{df(x, y)}{dx} & \frac{df(x, y)}{dy} \\ \frac{dg(x, y)}{dx} & \frac{dg(x, y)}{dy} \end{pmatrix}$$

Nonlinear Plane Autonomous Systems

THE LINEARIZATION PRINCIPLE

Fundamental Theorem of Stability

1. An equilibrium is *stable* if *all eigenvalues* of the linearization have *negative* real parts.
2. An equilibrium is *unstable* if at *least one eigenvalue* of the linearization has *positive* real part.

Nonlinear Plane Autonomous Systems

THE LINEARIZATION PRINCIPLE

Fundamental Theorem of Stability

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2. An equilibrium is *unstable* if at *least one eigenvalue* of the linearization has *positive* real part.

The Hartman/Grobman Theorem

If no eigenvalue has real part 0 (i.e., if the equilibrium is *hyperbolic*), then the *phase portrait* near the equilibrium is of the *same type* as that of the *linearization*.

EXAMPLE

$$x' = 3x - x^2 - xy$$

$$y' = 4y - xy - 2y^2$$

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$$x' = 3x - x^2 - xy$$

$$y' = 4y - xy - 2y^2$$

Equilibria

$$(x, y) = (0, 0), \quad (3, 0), \quad (0, 2), \quad (2, 1)$$

The Jacobian

$$J(x, y) = \begin{pmatrix} 3 - 2x - y & -x \\ -y & 4 - x - 4y \end{pmatrix}$$

EXAMPLE

$$x' = 3x - x^2 - xy$$

$$y' = 4y - xy - 2y^2$$

Linearization Analysis

$$(x, y) = (0, 0), \quad (3, 0), \quad (0, 2), \quad (2, 1)$$

$$J(0, 0) = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\lambda = 3, 4$$

unstable (node)

$$J(0, 2) = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$$

$$\lambda = 1, -4$$

unstable (saddle)

$$J(3, 0) = \begin{pmatrix} -3 & -3 \\ 0 & 1 \end{pmatrix}$$

$$\lambda = -3, 1$$

unstable (saddle)

USEFUL FACT: eigenvalues of a *triangular matrix* appear along the diagonal !

EXAMPLE

$$x' = 3x - x^2 - xy$$

$$y' = 4y - xy - 2y^2$$

Linearization Analysis

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$$\lambda = 1, -4$$

unstable (saddle)

$$J(3, 0) = \begin{pmatrix} -3 & -3 \\ 0 & 1 \end{pmatrix}$$

$$\lambda = -3, 1$$

unstable (saddle)

$$J(2, 1) = \begin{pmatrix} -2 & -2 \\ -1 & -2 \end{pmatrix}$$

$$\lambda = -2 \pm \sqrt{2} \approx -0.586, -3.414$$

stable (node)

Plot: Systems of ODEs



Commands Parameters **System** Window Auton

$du / dt =$

$dv / dt =$

$dw / dt =$

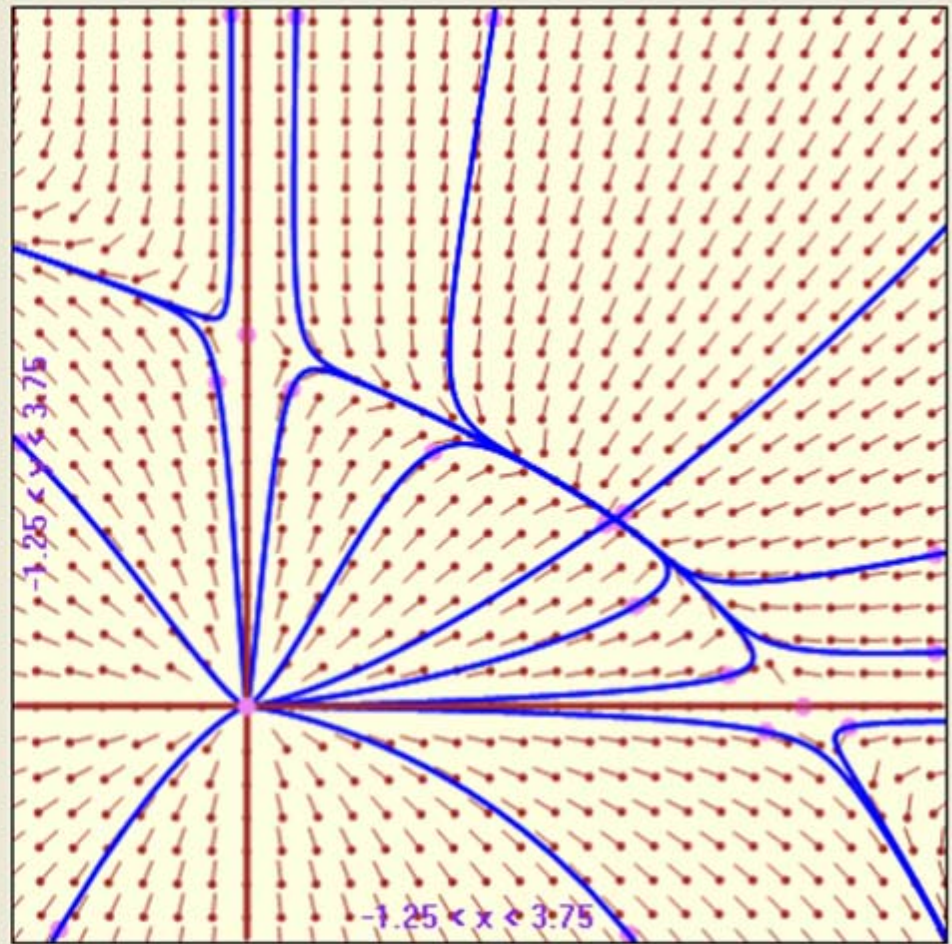
$dx / dt =$

$dy / dt =$

$dz / dt =$

$f(t) =$

y vs x



Time: ►

EXAMPLE

$$x'' + a(x^2 - 1)x' + x = 0 \quad (\text{van der Pol})$$

$$a = \text{constant}$$

$$-2 < a < 2$$

EXAMPLE

$$x' = y \quad (\text{van der Pol})$$

$$y' = -a(x^2 - 1)y - x$$

Equilibrium equations

$$y = 0$$

$$-a(x^2 - 1)y - x = 0$$

One equilibrium

$$(x, y) = (0, 0)$$

EXAMPLE

$$x' = y \quad (\text{van der Pol})$$

$$y' = -a(x^2 - 1)y - x$$

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -2ax - 1 & -a(x^2 - 1) \end{pmatrix} \Rightarrow J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$$

$$\lambda^2 - a\lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4} = \text{complex}$$

EXAMPLE

$$x' = y \quad (\text{van der Pol})$$

$$y' = -a(x^2 - 1)y - x$$

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ -2ax - 1 & -a(x^2 - 1) \end{pmatrix} \Rightarrow J(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$$

$$\lambda^2 - a\lambda + 1 = 0 \Rightarrow \lambda = \frac{1}{2}a \pm \frac{1}{2}\sqrt{a^2 - 4} = \text{complex}$$

$$a < 0 \Rightarrow (0, 0) \text{ is stable (spiral)}$$

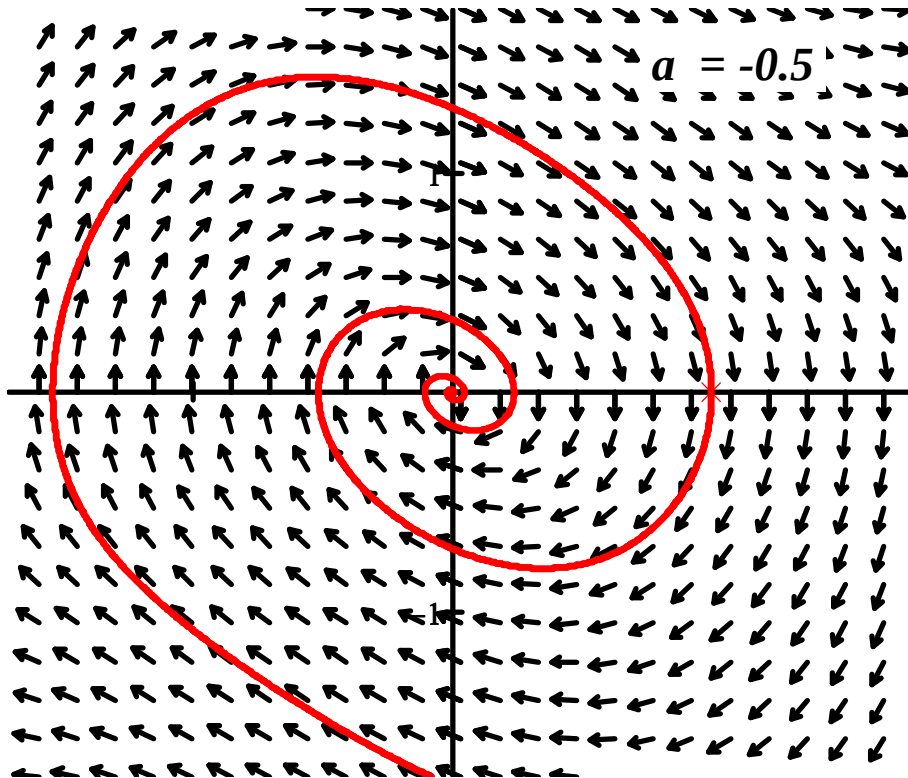
$$a > 0 \Rightarrow (0, 0) \text{ is unstable (spiral)}$$

$$a = 0 \Rightarrow (0, 0) \text{ non-hyperbolic}$$

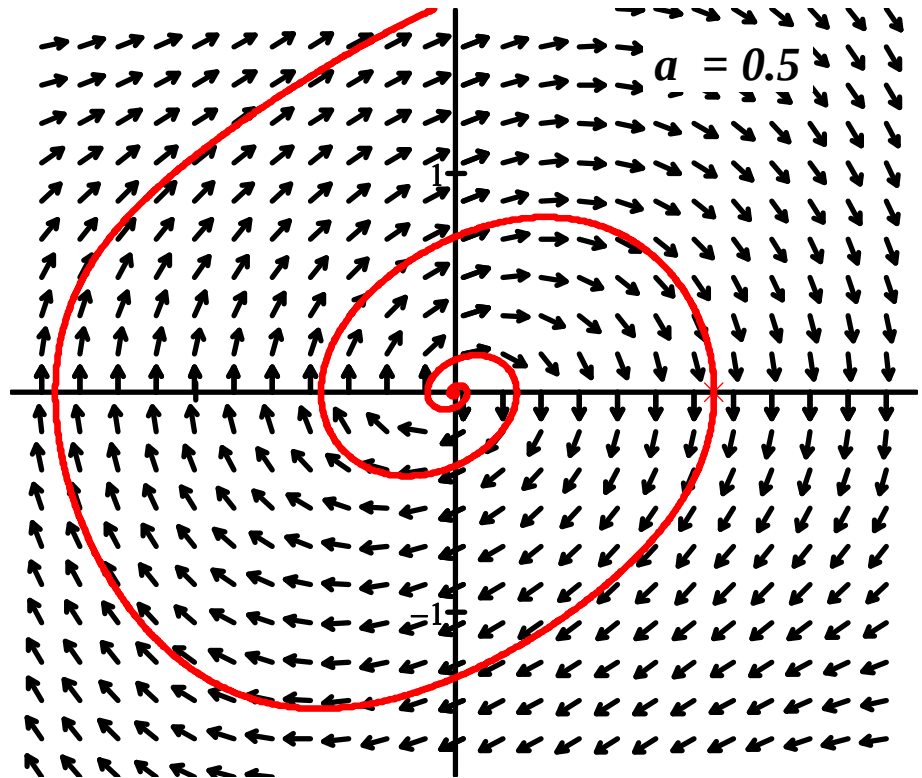
(linearization principle not applicable)

EXAMPLE

$$x'' + a(x^2 - 1)x' + x = 0 \quad (\text{van der Pol})$$



Stable Spiral



Unstable Spiral