

MATH 250b – Section 2

Spring Semester 2008

Thursday, 24-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

CHAPTER 8: Nonlinear Systems

Autonomous Systems

$$x' = f(x, y)$$

$$y' = g(x, y)$$

General problem is to determine the characteristics of the phase plane portrait

EXAMPLES

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

predator-prey

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

$$y' = 4 \frac{xy}{x + y} - 2y$$

HIV/AIDS

$$x'' + \alpha (x^2 - 1) x' + x = 0 \quad \text{van der Pol}$$

Linearization Analysis

First step: Find the equilibria (constant solution pairs)

Second step: Calculate the “linearization” at each equilibrium

Third step: Analyze the linearization (stability, phase portrait)

Fourth step: Draw appropriate conclusions about the nonlinear system

First step: find the equilibrium

The *equilibrium equations* are

$$f(x, y) = 0$$

$$g(x, y) = 0$$

A solution pair (x, y) of these algebraic equations
is an equilibrium

(a point orbit in the phase plane)

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

predator-prey

Equilibrium
equations

$$(y - 1)x = 0$$
$$(1 - x)y = 0$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

Second step: determine the phase plane portrait near each equilibrium

The main tool is the *Linearization Principle*

Basic Idea:

approximate the nonlinear system near an equilibrium by a *linear* system

The *linearization* of a nonlinear system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

at an equilibrium (x_e, y_e) is the linear system

$$\hat{x}' = A\hat{x}$$

where the coefficient matrix A is the *Jacobian Matrix*

$$J(x, y) = \begin{pmatrix} \frac{df(x, y)}{dx} & \frac{df(x, y)}{dy} \\ \frac{dg(x, y)}{dx} & \frac{dg(x, y)}{dy} \end{pmatrix}$$

evaluated at the equilibrium (x_e, y_e)

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(x, y) = \begin{pmatrix} \frac{df(x, y)}{dx} & \frac{df(x, y)}{dy} \\ \frac{dg(x, y)}{dx} & \frac{dg(x, y)}{dy} \end{pmatrix}$$

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(x, y) = \begin{pmatrix} y - 1 & x \\ -y & 1 - x \end{pmatrix}$$

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(x, y) = \begin{pmatrix} y - 1 & x \\ -y & 1 - x \end{pmatrix}$$

$$J(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

NOTE

The *linearization* at $(0, 0)$
has a *saddle*

and therefore is *unstable*

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(x, y) = \begin{pmatrix} y - 1 & x \\ -y & 1 - x \end{pmatrix}$$

$$J(1, 1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

NOTE

the *linearization* at $(1, 1)$
has a *center*

and therefore is *neither stable*
nor unstable
(it is called *neutrally stable*)

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

What can we learn from these linearizations about the phase portraits of the nonlinear system?

Theorem 1 (Linearization and stability)

Suppose (x_e, y_e) is an equilibrium of the system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

If at *both eigenvalues* of the Jacobian $J(x_e, y_e)$ satisfy $\operatorname{Re} \lambda < 0$,
then the equilibrium is *stable*.



All nearby orbits stay near & tend to the equilibrium as $t \rightarrow +\infty$.

Theorem 1 (Linearization and stability)

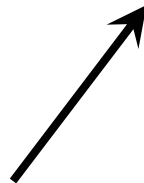
Suppose (x_e, y_e) is an equilibrium of the system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

If at *both eigenvalues* of the Jacobian $J(x_e, y_e)$ satisfy $\operatorname{Re} \lambda < 0$, then the equilibrium is *stable*.

If at *least one eigenvalue* of the Jacobian $J(x_e, y_e)$ satisfies $\operatorname{Re} \lambda > 0$, then the equilibrium is *unstable*.



Means: not stable

EXAMPLE

$$x' = (y - 1)x$$

$$y' = (1 - x)y$$

predator-prey

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = -1, 1 \Rightarrow (0, 0) \text{ is unstable}$$



positive

EXAMPLE

$$\begin{aligned}x' &= (y-1)x \\ y' &= (1-x)y\end{aligned}\quad \text{predator-prey}$$

$$(x_e, y_e) = (0, 0) \text{ and } (1, 1)$$

$$J(0, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \lambda = -1, 1 \Rightarrow (0, 0) \text{ is unstable}$$

$$J(1, 1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = \pm i \Rightarrow \text{theorem not applicable}$$

EXAMPLE

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

HIV/AIDS

$$y' = 4 \frac{xy}{x + y} - 2y$$

Equilibrium Equations

$$1 - x - 4 \frac{xy}{x + y} = 0$$

$$\left(4 \frac{x}{x + y} - 2 \right) y = 0$$

EXAMPLE

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

HIV/AIDS

$$y' = 4 \frac{xy}{x + y} - 2y$$

$$(x_e, y_e) = (1, 0) \text{ and } (1/3, 1/3)$$

$$J(x, y) = \begin{pmatrix} \frac{df(x, y)}{dx} & \frac{df(x, y)}{dy} \\ \frac{dg(x, y)}{dx} & \frac{dg(x, y)}{dy} \end{pmatrix}$$

EXAMPLE

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

HIV/AIDS

$$y' = 4 \frac{xy}{x + y} - 2y$$

$$(x_e, y_e) = (1, 0) \text{ and } (1/3, 1/3)$$

$$J(x, y) = \begin{pmatrix} -1 + \frac{4xy}{(x+y)^2} - \frac{4y}{x+y} & -\frac{4x}{x+y} + \frac{4xy}{(x+y)^2} \\ -\frac{4xy}{(x+y)^2} + \frac{4y}{x+y} & \frac{4x}{x+y} - \frac{4xy}{(x+y)^2} - 2 \end{pmatrix}$$

EXAMPLE

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

HIV/AIDS

$$y' = 4 \frac{xy}{x + y} - 2y$$

$$(x_e, y_e) = (1, 0) \text{ and } (1/3, 1/3)$$

$$J(1, 0) = \begin{pmatrix} -1 & -4 \\ 0 & 2 \end{pmatrix}$$

$$\lambda = -1, 2 \Rightarrow \text{unstable}$$

EXAMPLE

$$x' = 1 - x - 4 \frac{xy}{x + y}$$

HIV/AIDS

$$y' = 4 \frac{xy}{x + y} - 2y$$

$$(x_e, y_e) = (1, 0) \text{ and } (1/3, 1/3)$$

$$J(1/3, 1/3) = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\lambda = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i \Rightarrow \operatorname{Re} \lambda = -\frac{3}{2} < 0 \Rightarrow \text{stable}$$