

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 22-Apr-08

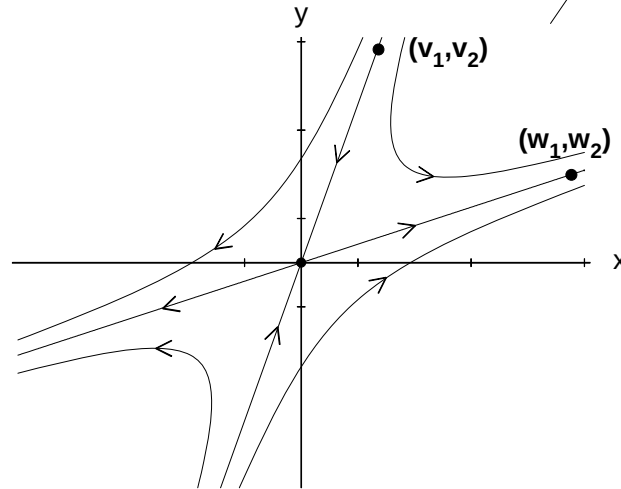
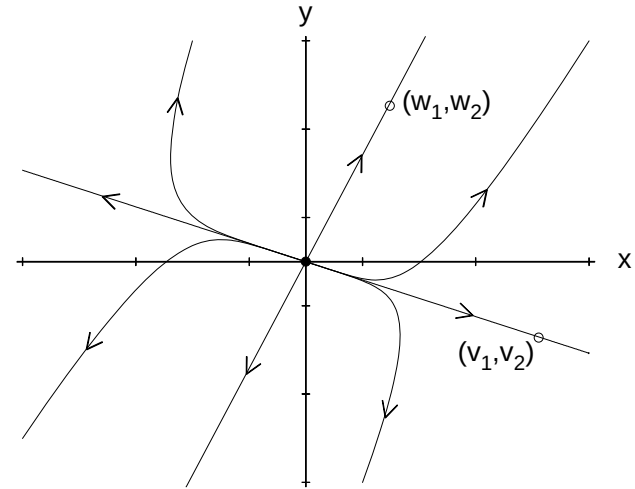
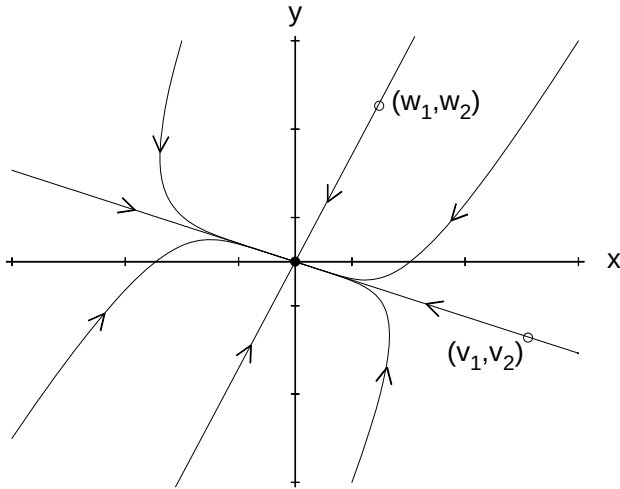
<http://math.arizona.edu/~cushing/250b.html>

Review

Case 1: two real and different roots $\lambda_1 \neq \lambda_2$

$\lambda_2 < \lambda_1 < 0$ *Node (stable)*

$0 < \lambda_1 < \lambda_2$ *Node (unstable)*



$\lambda_1 < 0 < \lambda_2$ *Saddle*

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

First case: $\alpha = 0$, $\beta = 1$

An Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = i, \quad \tilde{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

For example, if $c_1 = 1$, $c_2 = 0$ then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

Note: $x^2(t) + y^2(t) = 1$ for all t

Orbit lies on unit circle !

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

First case: $\alpha = 0$, $\beta = 1$

An Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = i, \quad \tilde{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

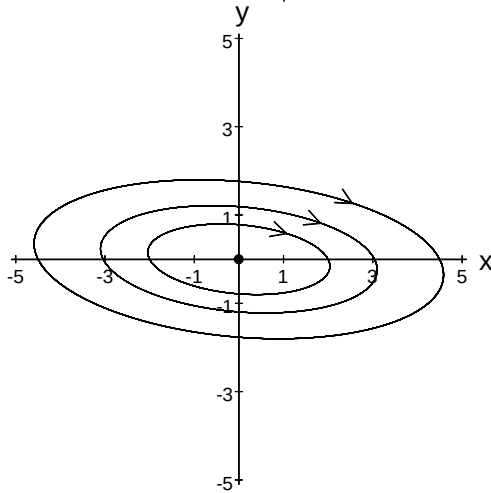
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

In general: $x^2(t) + y^2(t) = c_1^2 + c_2^2$ for all t

All orbits are circles !

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

$\alpha = 0$ *Center (neutrally stable)*



Orbits are ellipses centered on the origin

Plot a direction field arrow
to determine orientation
(clockwise or counter-clockwise)

EXAMPLE

$$A = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \Rightarrow \lambda_1 = i \Rightarrow \text{center}$$

Orbits are closed ellipses.

Are the clockwise or counter-clockwise ?

$$x' = -x + 2y$$

$$y' = -x + y$$

Draw a sample direction field arrow.

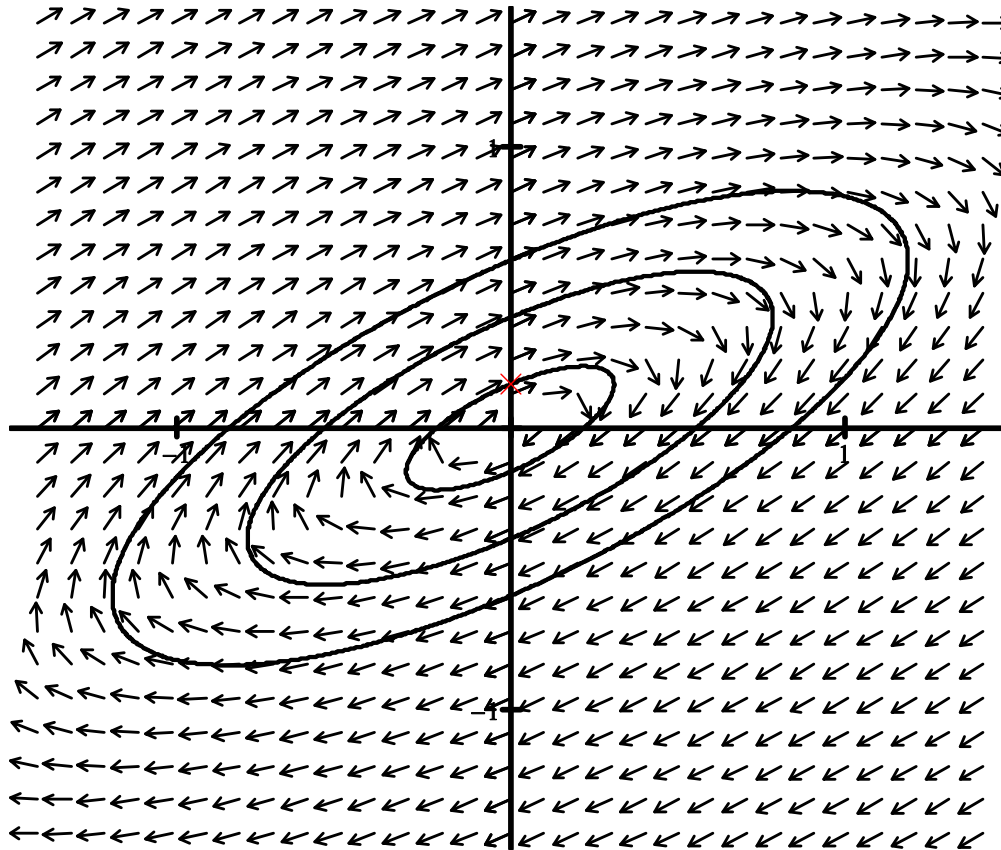
$$\text{For example, at point } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\text{we calculate } \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

which points SW $\Rightarrow$ orientation is clockwise.

EXAMPLE

$$A = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \Rightarrow \lambda_1 = i \Rightarrow \text{center}$$



Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Second case: $\alpha \neq 0$

An Example: $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda = -1 + i, \quad \hat{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \sin t & -e^{-t} \cos t \\ e^{-t} \cos t & e^{-t} \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \sin t - c_2 e^{-t} \cos t \\ c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \end{pmatrix}$$

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Case: $\alpha \neq 0$, $\beta \neq 0$

An Example: $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda = -1 + i, \quad \hat{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \sin t & -e^{-t} \cos t \\ e^{-t} \cos t & e^{-t} \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \sin t - c_2 e^{-t} \cos t \\ c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \end{pmatrix}$$

For example, if $c_1 = 1$, $c_2 = 0$ then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \cos t \\ e^{-t} \sin t \end{pmatrix}$$

Note: $x^2(t) + y^2(t) = e^{-2t}$ for all t

Orbit spirals into the origin !

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Case: $\alpha \neq 0$, $\beta \neq 0$

An Example: $A = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda = -1 + i, \quad \hat{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{-t} \sin t & -e^{-t} \cos t \\ e^{-t} \cos t & e^{-t} \sin t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \sin t - c_2 e^{-t} \cos t \\ c_1 e^{-t} \cos t + c_2 e^{-t} \sin t \end{pmatrix}$$

In general: $x^2(t) + y^2(t) = (c_1^2 + c_2^2) e^{-2t}$ for all t

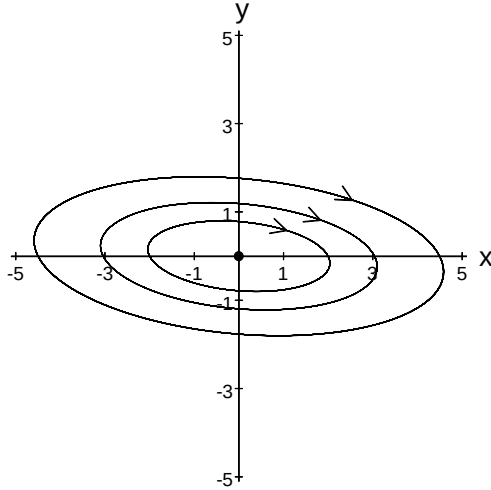
All orbits *spiral into the origin* !

NOTE: in this example $\alpha = -1 < 0$

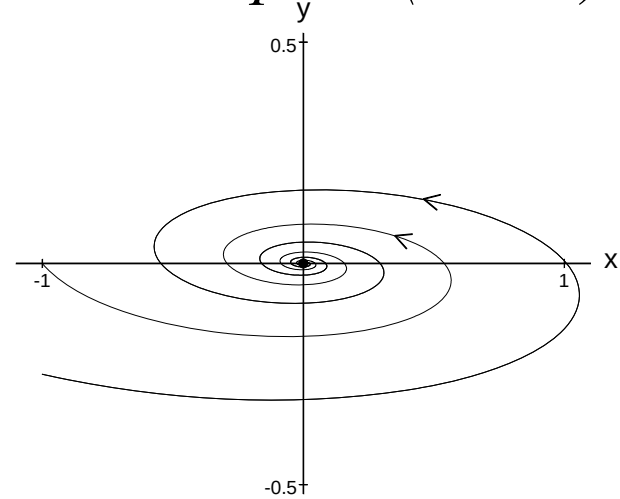
Real part of eigenvalue λ is negative

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

$\alpha = 0$ *Center (neutrally stable)*



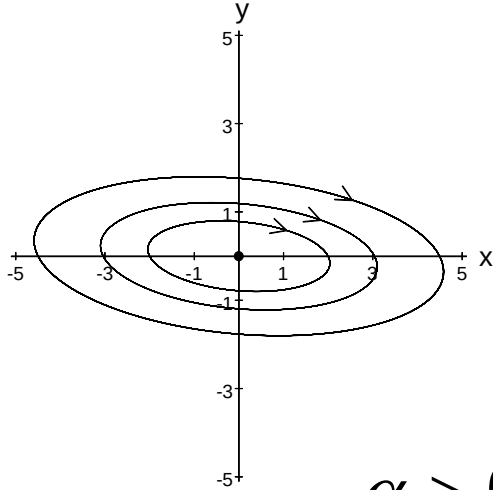
$\alpha < 0$ *Spiral (stable)*



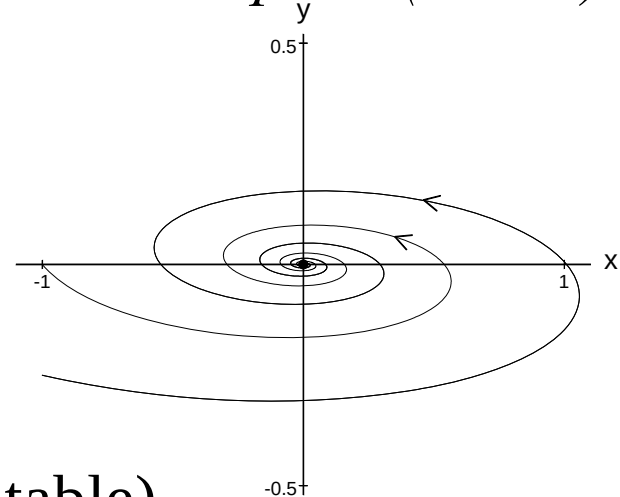
Plot a direction field arrow
to determine orientation
(clockwise or counter-clockwise)

Case 2: a complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

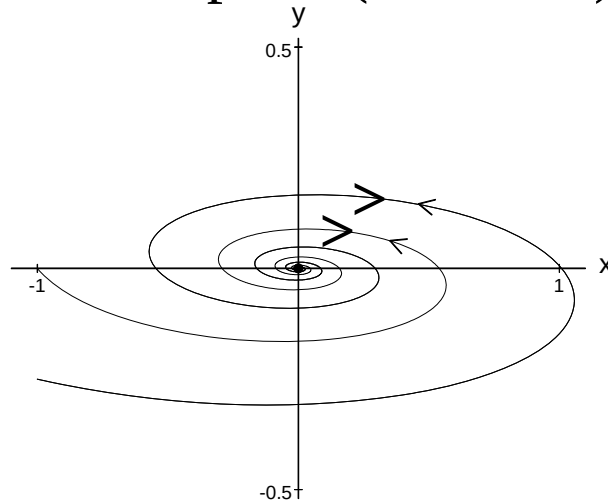
$\alpha = 0$ *Center (neutrally stable)*



$\alpha < 0$ *Spiral (stable)*



$\alpha > 0$ *Spiral (unstable)*



EXAMPLE

$$A = \begin{pmatrix} -1 & 5 \\ -4 & -1 \end{pmatrix} \Rightarrow \lambda_1 = -1 + 2\sqrt{5}i \Rightarrow \text{stable spiral}$$

Clockwise or counter-clockwise ?

Plot a convenient point in the direction field.

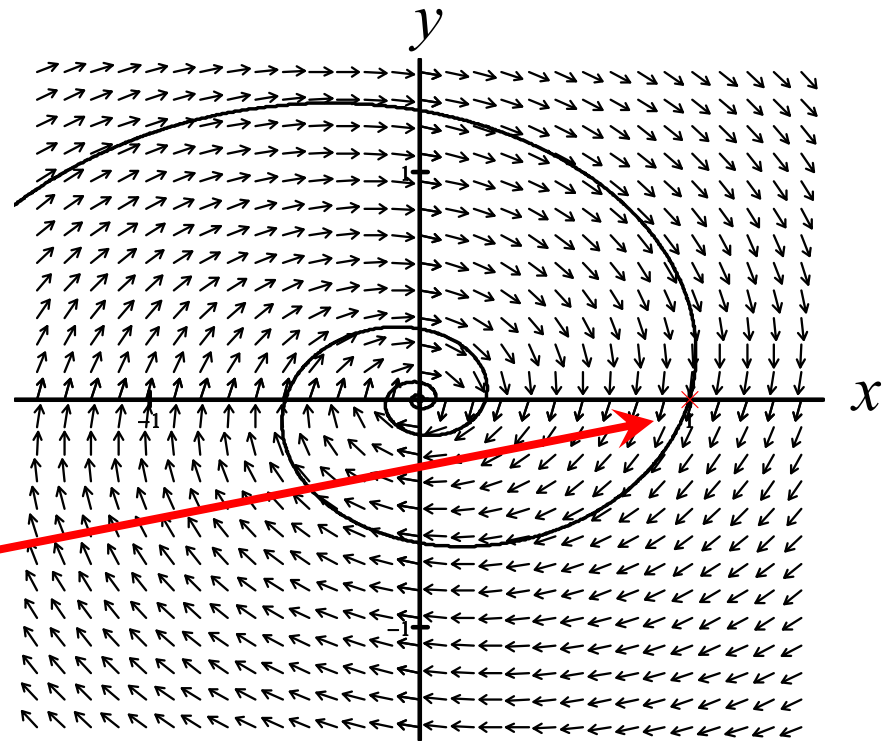
$$x' = -x + 5y$$

$$y' = -4x - y$$

For example, at $(x, y) = (1, 0)$:

$$(x', y') = (-1, -4)$$

points “southwest”
so spiral is *clockwise*



SUMMARY

Type

Stability

Case 1: node or saddle

stable if both $\lambda_i < 0$

unstable if at least one $\lambda_i > 0$

Case 2: spiral or center

stable if $\operatorname{Re}\lambda_i < 0$

unstable if $\operatorname{Re}\lambda_i > 0$

neutrally stable if $\operatorname{Re}\lambda_i = 0$

Case 3: degenerate node

stable if $\lambda < 0$

unstable if $\lambda > 0$

NOTES

- From the phase portrait summary, we see that
if the origin is *hyperbolic* (i.e., no $\text{Re } \lambda = 0$), then
it is *stable* if all **both** eigenvalues satisfy $\text{Re } \lambda < 0$
unstable if at **least one** eigenvalue satisfies $\text{Re } \lambda > 0$

That is to say, the stability of a phase portrait can be determined from the *sign of the real part* of the eigenvalues alone !

- Second order equation phase portraits have the same classifications.
(convert to a first order system)

Chapter 6: Nonhomogeneous Systems

$$\hat{x}' = A(t)\hat{x} + \hat{h}(t)$$

General Solution has the form

$$\hat{x}(t) = \hat{x}_h(t) + \hat{x}_p(t)$$

$$\hat{x}_h(t) = \Phi(t)\hat{c}$$

$\Phi(t)$ = fundamental solution matrix of $\hat{x}' = A(t)\hat{x}$

Problem: how to calculate a particular solution

$$\hat{x}_p(t) = ?$$

Two ways ...

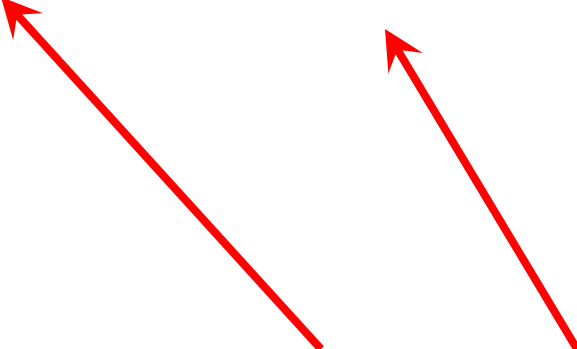
General method: Variation of Constants Formula

Shortcut: Method of Undetermined Coefficients

1. Variation of Constants Formula
(Theorem 1, p 248)

$$\hat{x}(t) = \Phi(t)\hat{c} + \Phi(t) \int \Phi^{-1}(t)\hat{h}(t)dt$$

Formula includes both x_h and x_p .



An Example

$$x' = -2x + 2y + e^{-2t}$$

$$y' = 2x - 5y$$

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}, \quad \hat{h}(t) = \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix}$$

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}$$

$$\hat{x}(t) = \Phi(t)\hat{c} + \Phi(t) \int \Phi^{-1}(t)\hat{h}(t)dt$$

$$\Phi^{-1}(t) = \begin{pmatrix} \frac{2}{5}e^t & \frac{1}{5}e^t \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix}$$

$$\Phi^{-1}(t)\tilde{h}(t) = \begin{pmatrix} \frac{2}{5}e^t & \frac{1}{5}e^t \\ \frac{1}{5}e^{6t} & -\frac{2}{5}e^{6t} \end{pmatrix} \begin{pmatrix} e^{-2t} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}e^{-t} \\ \frac{1}{5}e^{4t} \end{pmatrix}.$$

$$\int \Phi^{-1}(t)\tilde{h}(t)dt = \begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix}$$

$$\hat{x}(t) = \Phi(t)\hat{c} + \Phi(t) \int \Phi^{-1}(t)\hat{h}(t)dt$$

$$\begin{aligned}\Phi(t) \int^t \Phi^{-1}(s)\tilde{h}(s)ds &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} -\frac{2}{5}e^{-t} \\ \frac{1}{20}e^{4t} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{3}{4}e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} -\frac{3}{4}e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{pmatrix} \\ &= \begin{pmatrix} 2c_1e^{-t} + c_2e^{-6t} - \frac{3}{4}e^{-2t} \\ c_1e^{-t} - 2c_2e^{-6t} - \frac{1}{2}e^{-2t} \end{pmatrix}\end{aligned}$$

2. Method of Undetermined Coefficients (Section 2.3 & Exercises pp 64-65)

In this course, we'll use this method for second order equations.

$$ax'' + bx' + cx = h(t)$$

- Constant coefficients a , b , and c .
- Special “forcing” functions $h(t)$
 - exponentials
 - sines and cosines
 - polynomials
 - (and sums and products of these)
- *Create a guess* for x_p with *undetermined coefficients*,
which are then calculated by
substitution into the differential equation

The Method of Undetermined Coefficients

Steps (pages 64-65) :

1. Make sure the coefficients are constants.
2. Identify the (independent) functions that make up $h(t)$.
3. List all (independent) functions obtained by *repeated differentiation* of the functions in Step 2
4. If a solution of the homogeneous equation appears in the list, multiple the list by t
5. Form a linear combination of the list (with undetermined coefficients).
6. Substitute the linear combination into the differential equation to calculate its undetermined coefficients

Examples

$$x'' + 5x' + 6x = -6e^{-t}$$

$$\lambda^2 + 5\lambda + 6 = 0 \Rightarrow \lambda = -2, -3 \Rightarrow x_h = c_1 e^{-2t} + c_2 e^{-3t}$$

$h(t) = -6e^{-t}$ is a multiple of e^{-t}

The list of all derivatives of e^{-t} is just : e^{-t}

This list contains to solution from x_h .

Guess is: $x_p(t) = ke^{-t}$ (one undetermined coefficient k)

Substitute into the differential equation :

$$x'' + 5x' + 6x = ke^{-t} - 5ke^{-t} + 6ke^{-t} = 2ke^{-t}$$

To solve the differential equation we must take

$$2k = -6 \Rightarrow k = -3 \Rightarrow x_p(t) = -3e^{-t}$$

General solution : $x(t) = c_1 e^{-2t} + c_2 e^{-3t} - 3e^{-t}$

Examples

$$x'' + x = 3\sin 2t$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = i \Rightarrow x_h = c_1 \sin t + c_2 \cos t$$

$h(t) = 3\sin 2t$ is a multiple of $\sin 2t$

The list of all (independent) derivatives of $\sin 2t$ is : $\sin 2t, \cos 2t$

This list contains to solution from x_h .

Guess : $x_p(t) = k_1 \sin 2t + k_2 \cos 2t$ (undetermined coefficients k_1, k_2)

Substitute into the differential equation :

$$x'' + x = -3k_1 \sin 2t - 3k_2 \cos 2t$$

To solve the differential equation we must take

$$k_1 = -1, k_2 = 0 \Rightarrow x_p(t) = -\sin 2t$$

General solution : $x(t) = c_1 \sin t + c_2 \cos t - \sin 2t$

Example: Forced Harmonic Oscillator

$$mx'' + kx = \sin \beta t$$

NOTE

The *resonance case* corresponds to the case when a solution of the homogeneous equation appears in the guess for x_p when using the Method of Undetermined Coefficients & the list used to build the guess for x_p must be multiplied through by t . The solutions become unbounded in this case, which it turns out occurs when

$$\beta = \sqrt{k / m}.$$

When $\beta \neq \sqrt{k / m}$, solutions are bounded.