

MATH 250b – Section 2

Spring Semester 2008

Thursday, 17-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

Homogeneous Linear Autonomous 2nd Order Equations

$$ax'' + bx' + cx = 0, \quad a \neq 0$$

There is a *significant shortcut* for finding the general solution.

ROOTS OF EQUATION $a\lambda^2 + b\lambda + c = 0$	GENERAL SOLUTION OF $ax'' + bx' + cx = 0, \quad a \neq 0$
real & different $\lambda_1 \neq \lambda_2$	$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
complex conjugate $\lambda = \alpha \pm \beta i, \beta \neq 0$	$x = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)$
real & repeated $\lambda = \lambda_1 = \lambda_2$	$x = (c_1 + c_2 t) e^{\lambda t}$

ROOTS OF EQUATION**GENERAL SOLUTION OF**

$$a\lambda^2 + b\lambda + c = 0$$

$$ax'' + bx' + cx = 0, \quad a \neq 0$$

real & different $\lambda_1 \neq \lambda_2$

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Example

$$x'' - x' - 6x = 0$$

$$\lambda^2 - \lambda - 6 = 0$$

$$\lambda = -2, 3$$

$$x(t) = c_1 e^{-2t} + c_2 e^{3t}$$

ROOTS OF EQUATION**GENERAL SOLUTION OF**

$$a\lambda^2 + b\lambda + c = 0 \quad | \quad ax'' + bx' + cx = 0, \quad a \neq 0$$

$$\text{complex conjugate } \lambda = \alpha \pm \beta i, \beta \neq 0 \quad | \quad x = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$$

Example

$$x'' - 4x' + 13x = 0$$

$$\lambda^2 - 4\lambda + 13 = 0$$

$$\lambda = 2 \pm 3i$$

$$\alpha = 2, \quad \beta = 3$$

$$x(t) = c_1 e^{2t} \cos 3t + c_2 e^{2t} \sin 3t$$

ROOTS OF EQUATION**GENERAL SOLUTION OF**

$$a\lambda^2 + b\lambda + c = 0 \quad | \quad ax'' + bx' + cx = 0, \quad a \neq 0$$

$$\text{real \& repeated } \lambda = \lambda_1 = \lambda_2 \quad | \quad x = (c_1 + c_2 t)e^{\lambda t}$$

Example

$$4x'' - 4x' + x = 0$$

$$4\lambda^2 - 4\lambda + 1 = 0$$

$$\lambda = 1/2 \text{ (double root)}$$

$$x(t) = c_1 e^{t/2} + c_2 t e^{t/2} = (c_1 + c_2 t) e^{t/2}$$

Section 5.4 Phase Plane Portraits for autonomous homogeneous systems

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We only consider systems for which

$$\det A = ad - bc \neq 0$$

This means the *only constant (equilibrium) solution* is the trivial solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This solution's orbit is a point located in the phase plane at the origin.

This is the only point orbit.

EXAMPLE

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

Eigensolutions are

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}, \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$

What are their orbits in the phase plane ?

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What are their orbits in the phase plane ?

Components $x_1(t) = 2e^{-t}$, $y_1(t) = e^{-t}$ are positive and satisfy

$$y = \frac{1}{2}x$$

for all time t . The orbit is a half line !

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What are their orbits in the phase plane ?

Components $x_1(t) = e^{-6t}$, $y_1(t) = -2e^{-6t}$ are of one sign and satisfy

$$y = -2x$$

for all time t . The orbit is also a half line !

EXAMPLE

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

Multiple (nonzero) eigensolutions are

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t}, \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t}$$



Components $x_1(t) = 2c_1 e^{-t}$, $y_1(t) = c_1 e^{-t}$ are of one sign and satisfy

$$y = \frac{1}{2}x$$

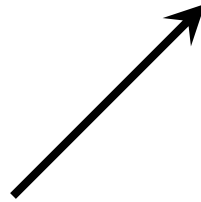
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Multiples (nonzero) of eigensolutions are

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Components $x_1(t) = c_1 e^{-6t}$, $y_1(t) = -2c_1 e^{-6t}$ are of one sign and satisfy

$$y = -2x$$

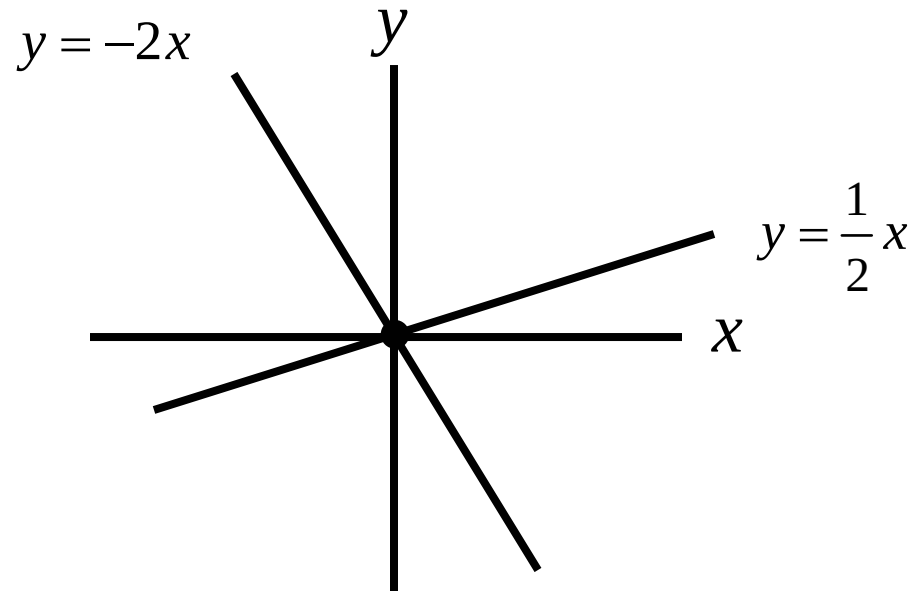
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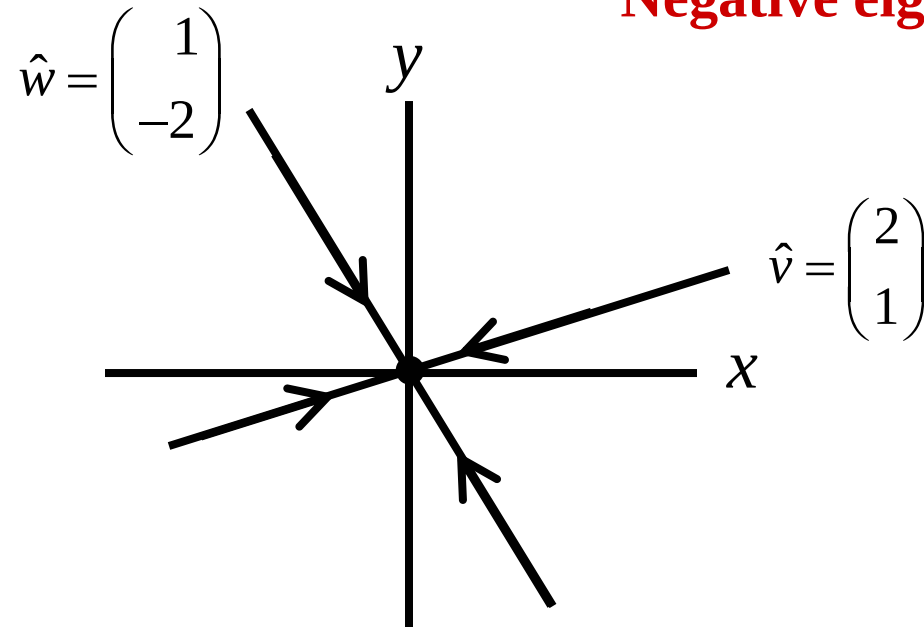
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Negative eigenvalues !

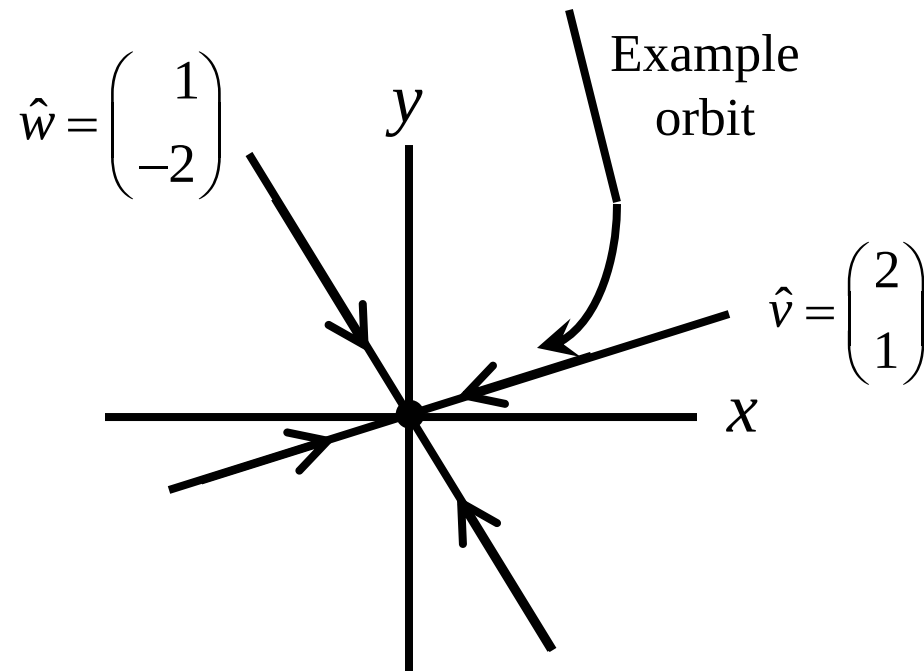


EXAMPLE

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

All other solutions are linear combinations

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} \approx c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} \text{ for large } t$$



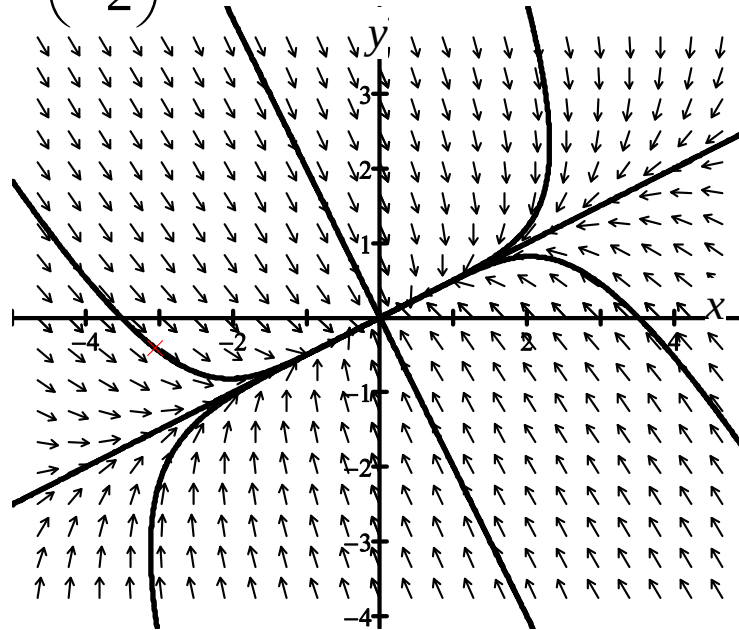
EXAMPLE

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

Computer Drawn Phase Plane Portrait

"fast time"
in direction of $\hat{w}$

$$\hat{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \lambda = -6$$



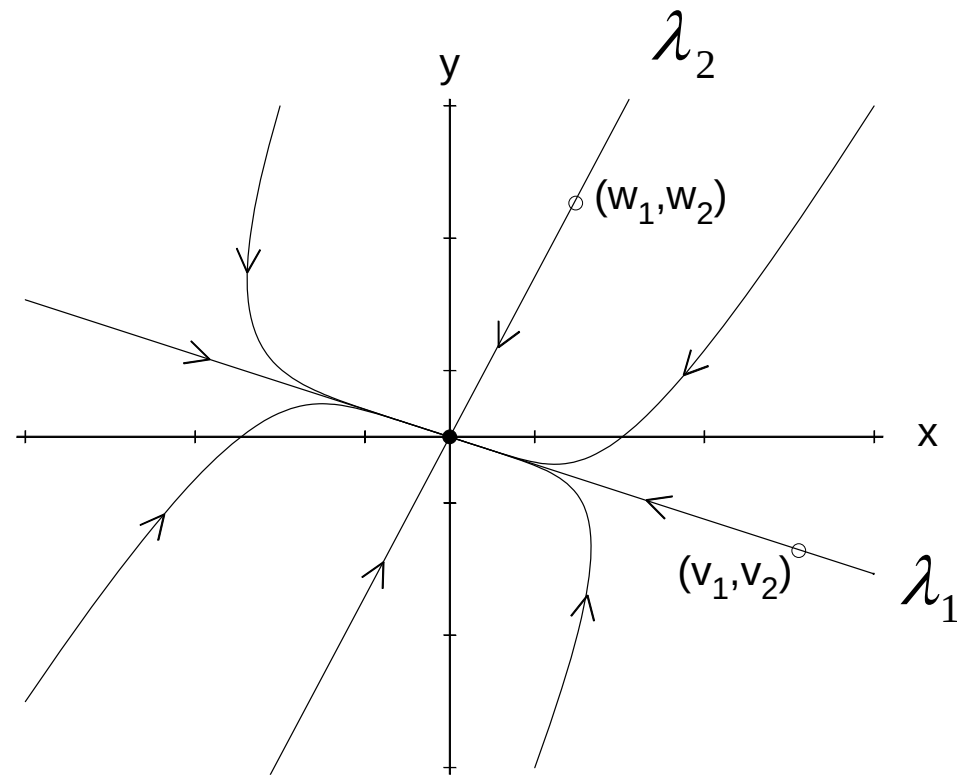
"slow time"
in direction of $\hat{v}$

$$\hat{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \lambda = -1$$

-1 is smaller
in magnitude !

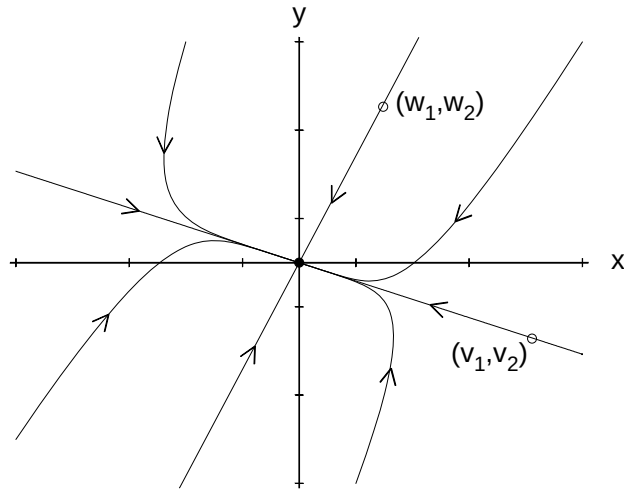
Case 1: two real and different roots $\lambda_1 \neq \lambda_2$

$\lambda_2 < \lambda_1 < 0$ *Node (stable)*

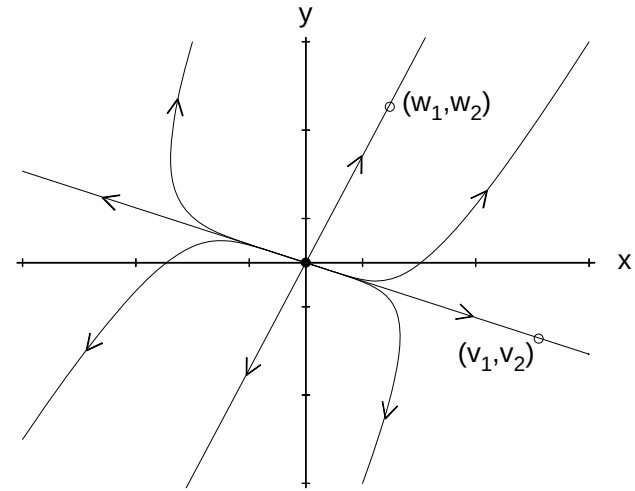


Case 1: two real and different roots $\lambda_1 \neq \lambda_2$

$\lambda_2 < \lambda_1 < 0$ Node (stable)



$0 < \lambda_1 < \lambda_2$ Node (unstable)



Orbits are “pinched” around eigenvector associated with eigenvalue of *smallest* magnitude.

EXAMPLE

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

$$\lambda_1 = -1, \quad \lambda_2 = -6 \quad \Rightarrow \quad \text{stable node}$$

$$\hat{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \hat{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$



NOTE:

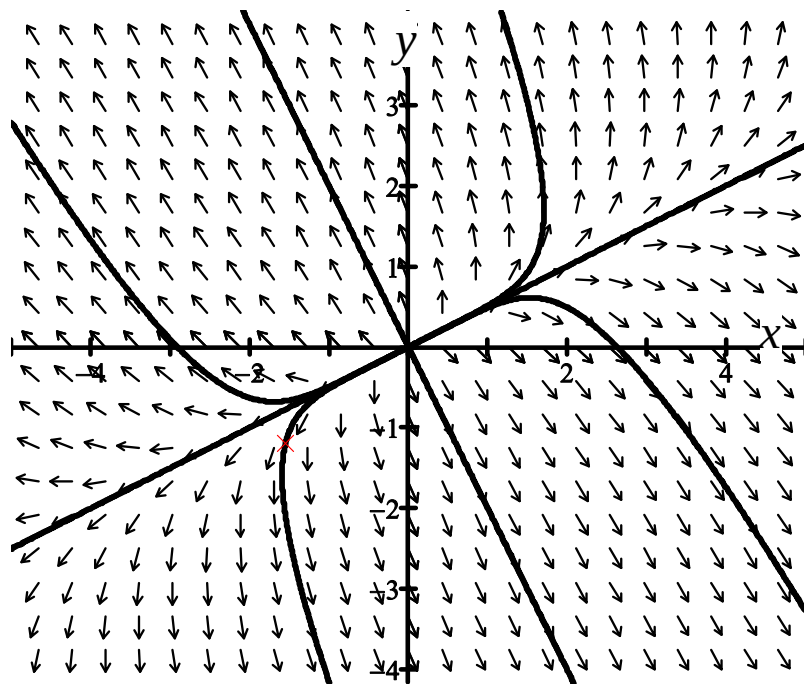
you can determine the
type & stability
from the eigenvalues alone

EXAMPLE

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$$

$$\lambda_1 = 1, \quad \lambda_2 = 6 \quad \Rightarrow \quad \text{unstable node}$$

$$\hat{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \hat{w} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$



What about when $\lambda_1 < 0 < \lambda_2$?

EXAMPLE

$$A = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix}$$

$$\lambda_1 = -3, \quad \lambda_2 = 2$$

$$\hat{v} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}, \quad \hat{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \approx \begin{cases} c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} & \text{for } t \rightarrow +\infty \\ c_1 \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-3t} & \text{for } t \rightarrow -\infty \end{cases}$$

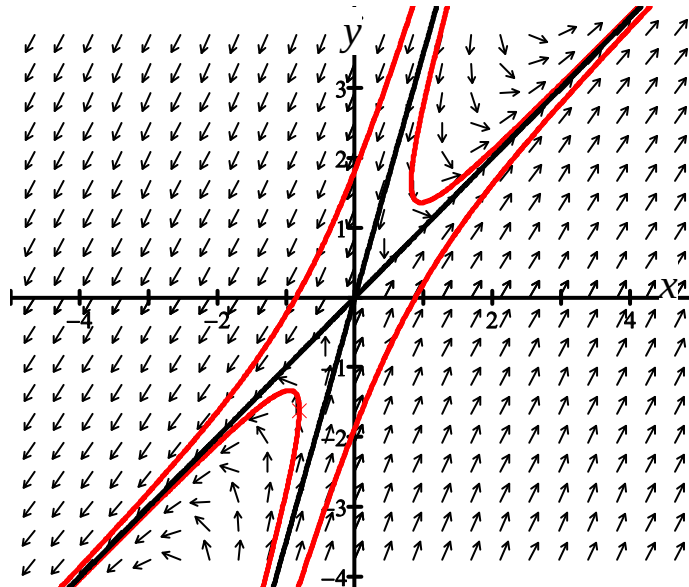
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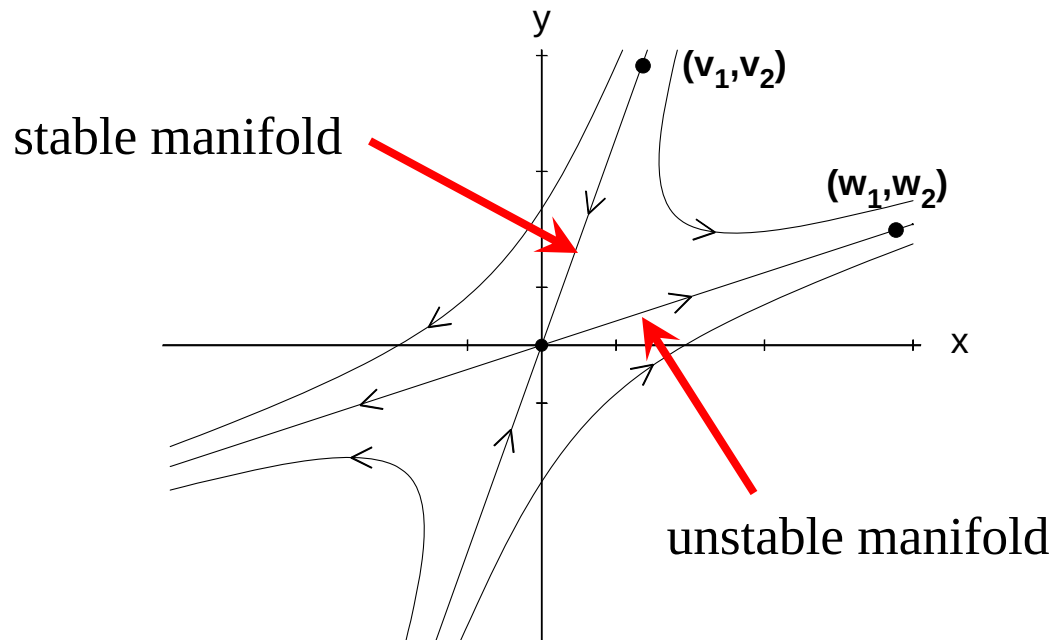
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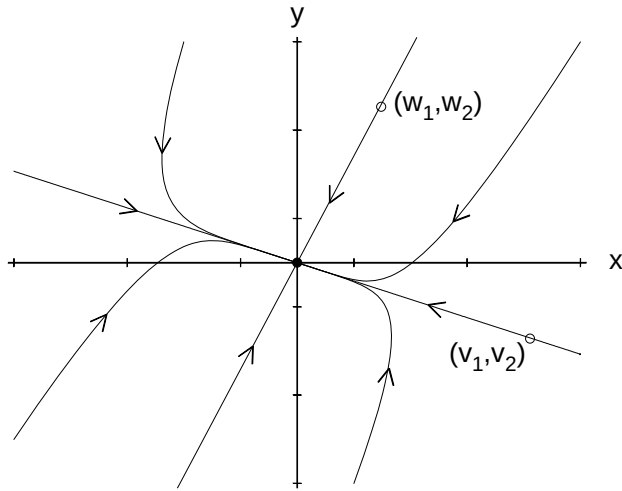
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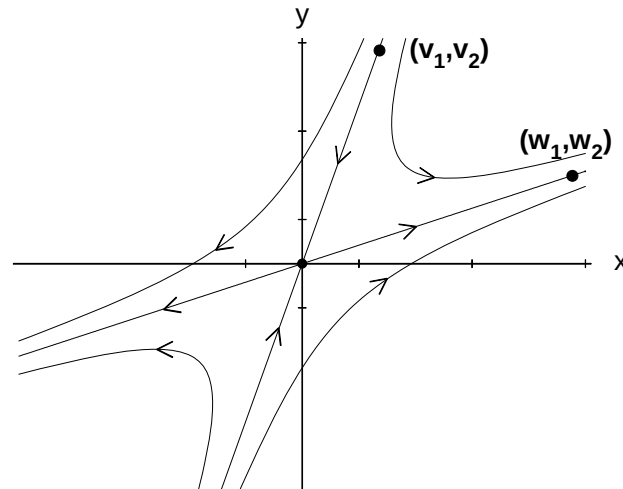
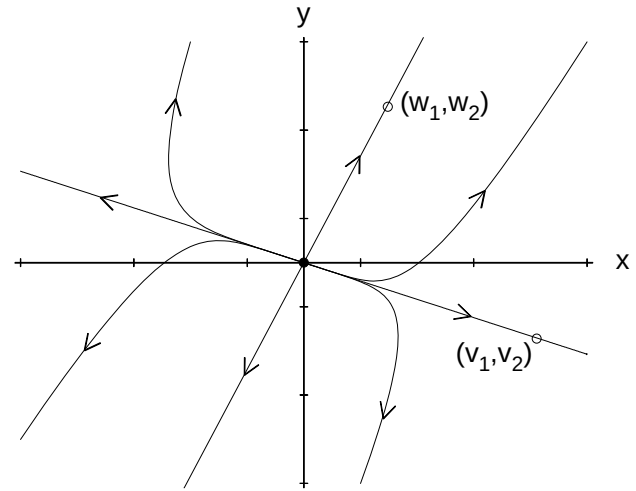
$\lambda_1 < 0 < \lambda_2$ Saddle

Case 1: two real and different roots $\lambda_1 \neq \lambda_2$

$\lambda_2 < \lambda_1 < 0$ *Node (stable)*



$0 < \lambda_1 < \lambda_2$ *Node (unstable)*



$\lambda_1 < 0 < \lambda_2$ *Saddle*