

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 15-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

Review: How to Calculate Eigenvalues & Vectors

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - (a + d)\lambda + (ad - bc)$$

Eigenvector components v_1, v_2 are solutions
of the (dependent) equations

$$(a - \lambda)v_1 + bv_2 = 0$$

$$cv_1 + (d - \lambda)v_2 = 0$$

Review

To calculate general solution of a homogeneous, autonomous linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- Calculate the eigenvalues & eigenvectors of the coefficient matrix

$$\lambda_1 \text{ and } \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \lambda_2 \text{ and } \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

- Build two “eigen-solutions” $\hat{x}_1 = \begin{pmatrix} v_1 e^{\lambda_1 t} \\ v_2 e^{\lambda_1 t} \end{pmatrix}, \quad \hat{x}_2 = \begin{pmatrix} w_1 e^{\lambda_2 t} \\ w_2 e^{\lambda_2 t} \end{pmatrix}$

- Build a Fundamental Solution Matrix $\Phi(t) = \begin{pmatrix} v_1 e^{\lambda_1 t} & w_1 e^{\lambda_2 t} \\ v_2 e^{\lambda_1 t} & w_2 e^{\lambda_2 t} \end{pmatrix}$

- General solution is $\Phi(t)\hat{c} = \begin{pmatrix} v_1 e^{\lambda_1 t} & w_1 e^{\lambda_2 t} \\ v_2 e^{\lambda_1 t} & w_2 e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 v_1 e^{\lambda_1 t} + c_2 w_1 e^{\lambda_2 t} \\ c_1 v_2 e^{\lambda_1 t} + c_2 w_2 e^{\lambda_2 t} \end{pmatrix}$

—————→ Works when *eigenvalues are real and distinct*.

An Application

Step 1

$$x' = -\alpha x - \beta y$$

$$y' = \gamma x - \delta y$$

$x(t)$ = excess glucose in the blood stream

$y(t)$ = excess insulin in the blood stream

An Application

Step 1

$$x' = -\alpha x - \beta y$$

$$y' = \gamma x - \delta y$$

$x(t)$ = excess glucose in the blood stream

$y(t)$ = excess insulin in the blood stream

rate liver absorbs glucose

rate muscles absorb glucose

An Application

Step 1

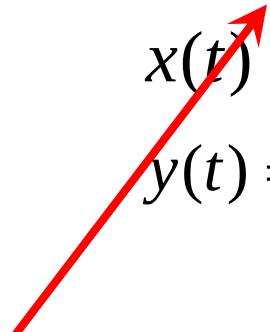
$$x' = -\alpha x - \beta y$$

$$y' = \gamma x - \delta y$$

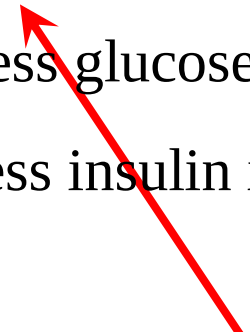
$x(t)$ = excess glucose in the blood stream

$y(t)$ = excess insulin in the blood stream

rate pancreas
produces insulin



rate liver degrades insulin



An Application

Step 1

$$x' = -\alpha x - \beta y$$

$$y' = \gamma x - \delta y$$

$x(t)$ = excess glucose in the blood stream

$y(t)$ = excess insulin in the blood stream

Experimentally estimated parameter values:

$$\alpha = 2.92, \quad \beta = 4.34, \quad \gamma = 0.208, \quad \delta = 0.780$$

Coefficient matrix is

$$A = \begin{pmatrix} -\alpha & -\beta \\ \gamma & -\delta \end{pmatrix} = \begin{pmatrix} -2.92 & -4.34 \\ 0.208 & -0.780 \end{pmatrix}$$

An Application

Step 2

Coefficient matrix is

$$A = \begin{pmatrix} -2.92 & -4.34 \\ 0.208 & -0.780 \end{pmatrix}$$

Characteristic polynomial

$$\lambda^2 - \text{tr}A \lambda + \det A$$

$$\lambda^2 + 3.7\lambda + 3.1803$$

Eigenvalues

$$\lambda_1 = -2.34, \quad \lambda_2 = -1.36$$

Eigenvalues & Eigenvectors

$$\lambda_1 = -2.34, \quad \hat{v} = \begin{pmatrix} 4.34 \\ -0.58 \end{pmatrix}$$

$$\lambda_2 = -1.36, \quad \hat{w} = \begin{pmatrix} 4.34 \\ -1.56 \end{pmatrix}$$

Eigen-solutions

$$\hat{x}_1 = \begin{pmatrix} 4.34e^{-2.34t} \\ -0.58e^{-2.34t} \end{pmatrix}$$

$$\hat{x}_2 = \begin{pmatrix} 4.34e^{-1.36t} \\ -1.56e^{-1.36t} \end{pmatrix}$$

An Application

Step 2

General Solution :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 4.34e^{-2.34t} & 4.34e^{-1.36t} \\ -0.58e^{-2.34t} & -1.56e^{-1.36t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$
$$= \begin{pmatrix} 4.34c_1e^{-2.34t} + 4.34c_2e^{-1.36t} \\ -0.58c_1e^{-2.34t} - 1.56c_2e^{-1.36t} \end{pmatrix}$$

Step 3

$$\text{NOTE: } \lim_{t \rightarrow +\infty} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

*Glucose & insulin levels
return (exponentially) to equilibrium levels*

An Application

Step 1

Liver failure: $\alpha = 0$

$$x' = -\beta y$$

$$y' = \gamma x - \delta y$$

$$\beta = 4.34, \quad \gamma = 0.208, \quad \delta = 0.780$$

Step 2

Characteristic polynomial

$$\lambda^2 + 0.78\lambda + 0.91$$

has roots

$$\lambda = -0.39 \pm 0.87i = \text{complex !}$$

Algebraically there are three cases:

The characteristic polynomial has

Case 1: two real and different roots $\lambda_1 \neq \lambda_2$

Case 2: complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Case 3: a double (repeated) root $\lambda_1 = \lambda_2 = \lambda$

Some Review Facts about Complex Numbers

Notation: $\alpha + i\beta = re^{i\theta}$

Cauchy-Euler Formula:

$$\cos \theta + i \sin \theta = e^{i\theta}$$

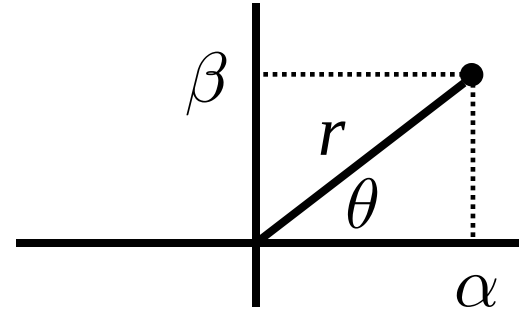
Multiplication:

In standard form, multiply as usual, remember that $i^2 = -1$

$$(-1 + 2i)(2 - 3i) = [-2 + 6] + i[3 + 4] = 4 + 7i$$

In exponential form, use usual rules for exponents

$$(2e^{5i})(4e^{-2i}) = 8e^{5i-2i} = 8e^{3i}$$



$$r = \sqrt{\alpha^2 + \beta^2},$$

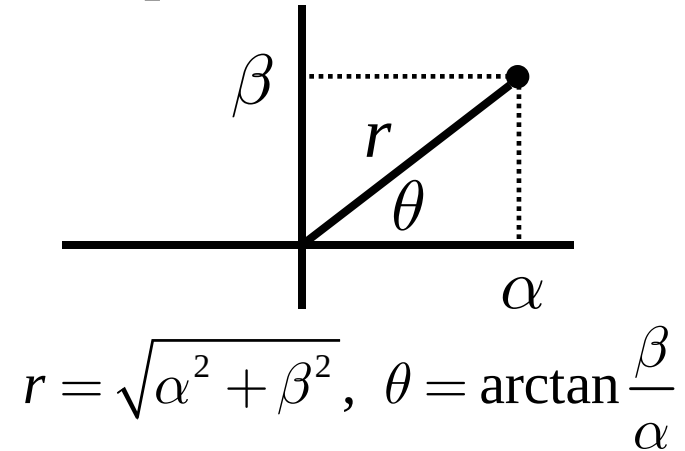
$$\sin \theta = \beta / r, \quad \cos \theta = \alpha / r$$

Some Review Facts about Complex Numbers

Notation: $\alpha + i\beta = re^{i\theta}$

Cauchy-Euler Formula:

$$\cos \theta + i \sin \theta = e^{i\theta}$$



Multiplication:

Mixed forms: pick a common form and convert

$$\begin{aligned} (\sqrt{3} + i)e^{it} &= (\sqrt{3} + i)(\cos t + i \sin t) \\ &= [\sqrt{3} \cos t - \sin t] + i[\sqrt{3} \sin t + \cos t] \end{aligned}$$

or

$$(\sqrt{3} + i)e^{it} = (2e^{\pi i/6})e^{it} = 2e^{(t+\pi/6)i}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Case 2: complex (conjugate) roots $\lambda = \alpha \pm i\beta$, $\beta \neq 0$

Find an eigenvector for $\lambda = \alpha + i\beta$ and build exponential solution:

$$\hat{v}e^{\lambda t} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \text{complex}$$

The real and imaginary parts will be independent solutions.

$$\hat{v}e^{\lambda t} = \begin{pmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + i \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

General Solution: $\Phi(t)\hat{c} = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Real part

Imaginary part

Example (Harmonic Oscillator)

$$x'' + x = 0 \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

Characteristic polynomial: $\lambda^2 + 1 = 0$

Complex eigenvalues: $\lambda = \pm i$ ($\alpha = 0$ and $\beta = 1$)

Complex eigenvector: $\hat{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Complex solution: $\hat{v}e^{\lambda t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

General Solution: $\Phi(t)\hat{c} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

or ... $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$

Example (Harmonic Oscillator)

$$x'' + x = 0 \quad \Longleftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

Note that periodic oscillations occur in this example.

It is typical of Case 2 that oscillations occur in the general solution
(although not necessarily periodic, as we will see)

Example

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & -6 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Characteristic polynomial $\lambda^2 - 2\lambda + 10$ has complex roots $\lambda = 1 \pm 3i$.

An eigenvector for $\lambda = 1 + 3i$ is $\hat{v} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$.

Complex solution: $\hat{v}e^{\lambda t} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} e^{(1+3i)t} = e^t \begin{pmatrix} \cos 3t - \sin 3t \\ \cos 3t \end{pmatrix} + ie^t \begin{pmatrix} \cos 3t + \sin 3t \\ \sin 3t \end{pmatrix}$

General Solution: $\Phi(t)\hat{c} = \begin{pmatrix} e^t (\cos 3t - \sin 3t) & e^t (\cos 3t + \sin 3t) \\ e^t \cos 3t & e^t \sin 3t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

or ... $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t (\cos 3t - \sin 3t) c_1 + e^t (\cos 3t + \sin 3t) c_2 \\ c_1 e^t \cos 3t + c_2 e^t \sin 3t \end{pmatrix}$

Example

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & -6 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t (\cos 3t - \sin 3t) c_1 + e^t (\cos 3t + \sin 3t) c_2 \\ c_1 e^t \cos 3t + c_2 e^t \sin 3t \end{pmatrix}$$

Note that *undamped oscillations* occur in this example.

Case 2

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

complex eigenvalue $\lambda = \alpha + i\beta$, $\beta \neq 0$

$$\hat{v}e^{\lambda t} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \text{complex eigenvector}$$

$$e^{\lambda t} = e^{\alpha t} (\cos \beta t + i \sin \beta t)$$

$$\text{Both } \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \text{ and } \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t} [\text{linear combo of } \cos \beta t, \sin \beta t] \\ e^{\alpha t} [\text{linear combo of } \cos \beta t, \sin \beta t] \end{pmatrix}$$

undamped oscillations occur if $\alpha > 0$

undamped oscillations occur if $\alpha < 0$

periodic oscillations occur if $\alpha = 0$

β determines the period $2\pi/\beta$
of the (shifted) periodic oscillations
in the square brackets [...]

An Application

Step 1

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$y(t)$ = excess insulin in the blood stream

Experimentally estimated parameter values:

$$\alpha = 2.92, \quad \beta = 4.34, \quad \gamma = 0.208, \quad \delta = 0.780$$

Liver failure: $\alpha = 0$

Step 2 $\begin{pmatrix} 0 & -4.34 \\ 0.208 & -0.780 \end{pmatrix}$ has complex root: $\lambda = -0.39 \pm 0.87i$

$\alpha = -0.39 < 0 \Rightarrow$ damped oscillations with period $= 2\pi / \beta \approx 7.2$ (time units)

Step 3

Blood glucose & insulin levels return to normal,
but in an *oscillatory* fashion

