

MATH 250b – Section 2

Spring Semester 2008

Thursday, 10-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

Homogeneous Linear Systems

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The general solution has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

where $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are two *independent* solution pairs



means

“are not multiples of each other”

A Useful Test for Independence

Two solution pairs

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}, \quad \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

are independent *if and only if*

$$\det \Phi(t) = \det \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix} \neq 0 \text{ for all } t$$

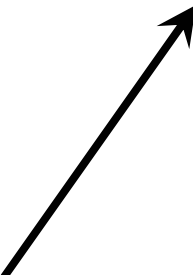
$$\text{RECALL: } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

$$\hat{x}_1 = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix}$$

$$\det \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} = -4e^{-7t} - e^{-7t} = -5e^{-7t} \neq 0$$

$$\begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix} \text{ are independent}$$


Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

General
Solution

$$\hat{x}(t) = c_1 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix} = \begin{pmatrix} 2c_1e^{-t} + c_2e^{-6t} \\ c_1e^{-t} - 2c_2e^{-6t} \end{pmatrix}$$

Examples

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$$\hat{x}(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

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Solution

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$$\hat{x}(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

General Solution:

$$\hat{x}(t) = \Phi(t)\hat{c}$$

Fundamental
Solution
Matrix

$$\Phi(t) = \begin{pmatrix} 2e^{-t} & e^{-6t} \\ e^{-t} & -2e^{-6t} \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

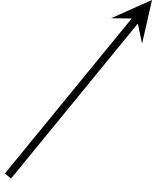
Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\hat{x}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$$

$\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ are independent



Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

General Solution:

$$\hat{x}(t) = \Phi(t)\hat{c}$$

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\hat{x}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

Linear Homogeneous Systems

$$\hat{x}' = A(t)\hat{x} \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$$

GENERAL SOLUTION : $\hat{x}(t) = \Phi(t)\hat{c}$

$$\Phi(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ y_1(t) & y_2(t) \end{pmatrix}, \quad \hat{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Fundamental solution matrix
(columns are independent solutions)

Vector of
arbitrary constants

PROBLEM: How does one find two independent solutions ?

Chapter 5: Homogeneous Linear Systems

GOAL: To find formulas for two independent solutions of an autonomous systems

$$\hat{x}' = A\hat{x}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{constant}$$

$$\hat{x} = \hat{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is called the } \textit{trivial solution}$$

It cannot be used in a fundamental solution matrix.

Some Facts about Linear Algebraic Systems of Equations

$$ax + by = h_1$$

$$cx + dy = h_2$$

- There is either *no* solution, *one* solution, or *infinitely many* solutions
(Think intersecting straight lines)

Some Facts about Linear Algebraic Systems of Equations

$$ax + by = h_1$$

$$cx + dy = h_2$$

- There is either *no* solution, *one* solution, or *infinitely many* solutions
(Think intersecting straight lines)
-

$$ax + by = 0$$

$$cx + dy = 0$$

- $x = 0, y = 0$ is a solution (the "trivial" solution)
- $x = 0, y = 0$ is the only solution if and only if

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

(equations are dependent)

Method is based on a search for exponential solutions.

Example

$$\hat{x}' = A(t)\hat{x}, \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \quad \begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned}$$

Find v_1, v_2 and λ so that

$$\begin{aligned} x(t) &= v_1 e^{\lambda t} \\ y(t) &= v_2 e^{\lambda t} \end{aligned} \quad \text{is a solution pair.}$$

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned} \Rightarrow \begin{aligned} \lambda v_1 e^{\lambda t} &= -2v_1 e^{\lambda t} + 2v_2 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} &= 2v_1 e^{\lambda t} - 5v_2 e^{\lambda t} \end{aligned}$$

$$\Rightarrow \begin{aligned} \lambda v_1 &= -2v_1 + 2v_2 \\ \lambda v_2 &= 2v_1 - 5v_2 \end{aligned}$$

v_1, v_2 and λ must solve

$$\begin{aligned} (-2 - \lambda)v_1 + 2v_2 &= 0 \\ 2v_1 + (-5 - \lambda)v_2 &= 0 \end{aligned}$$

Don't want $v_1 = 0, v_2 = 0$.

Must choose λ so equations are dependent:

$$(-2 - \lambda)(-5 - \lambda) - 2 \cdot 2 = 0$$

$$\lambda^2 + 7\lambda + 6 = 0$$

$$\lambda = -1, -6 \longleftarrow \text{Two choices}$$

$$(-2 - \lambda)v_1 + 2v_2 = 0$$

$$2v_1 + (-5 - \lambda)v_2 = 0$$

$$x(t) = v_1 e^{\lambda t}$$

$$y(t) = v_2 e^{\lambda t}$$

$$\lambda = -1$$

$$-v_1 + 2v_2 = 0$$

A choice is

$$v_1 = 2, \quad v_2 = 1$$

$$\Rightarrow \hat{x}_1 = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix}$$

$$\lambda = -6$$

$$4v_1 + 2v_2 = 0$$

A choice is

$$v_1 = 1, \quad v_2 = -2$$

$$\Rightarrow \hat{x}_2 = \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix}$$

Re-write these calculations using matrix notation and algebra :

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}$$

$$\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}$$

 cancel $e^{\lambda t}$

$$\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

λ is an eigenvalue of $\begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$ and $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector

Compact Matrix Notation

Method works in general

$$\hat{x}' = A\hat{x}, \quad A = \text{constant}$$

Find a vector $\hat{v} \neq \hat{0}$ and a number λ so that

$$\hat{x}(t) = \hat{v} e^{\lambda t} \text{ is a solution.}$$

Substitute $\hat{x}(t) = \hat{v} e^{\lambda t}$ and cancel $e^{\lambda t}$:

$$\lambda \hat{v} e^{\lambda t} = A \hat{v} e^{\lambda t}$$

$$A \hat{v} = \lambda \hat{v}, \quad \hat{v} \neq \hat{0}$$

λ must be an eigenvalue, and $\hat{v}$ is an eigenvector, of A

$\hat{x}(t) = \hat{v} e^{\lambda t}$ is then an *eigen-solution*.

Review of How to Calculate Eigenvalues & Vectors

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Eigenvalues are the roots of the characteristic polynomial

$$\lambda^2 - (a + d)\lambda + (ad - bc)$$

Eigenvector components v_1, v_2 are solutions
of the (dependent) equations

$$(a - \lambda)v_1 + bv_2 = 0$$

$$cv_1 + (d - \lambda)v_2 = 0$$

Example

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}$$

The characteristic polynomial of the coefficient matrix is

$$\lambda^2 + \lambda - 6$$

The roots (eigenvalues) are $\lambda = 2, -3$

$$\lambda = 2$$

$$2v_1 - 2v_2 = 0$$

$$7v_1 - 7v_2 = 0$$

A choice is
 $v_1 = 1, v_2 = 1 \Rightarrow \hat{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$

$$\lambda = -3$$

$$7v_1 - 2v_2 = 0$$

$$7v_1 - 2v_2 = 0$$

A choice is
 $v_1 = 2, v_2 = 7 \Rightarrow \hat{x}_2 = \begin{pmatrix} 2 \\ 7 \end{pmatrix} e^{-3t}$

Example

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 4 & -2 \\ 7 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} e^{\lambda t}$$

Fundamental
solution matrix

$$\Phi(t) = \begin{pmatrix} e^{2t} & 2e^{-3t} \\ e^{2t} & 7e^{-3t} \end{pmatrix}$$

General solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} & 2e^{-3t} \\ e^{2t} & 7e^{-3t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{2t} + 2c_2 e^{-3t} \\ c_1 e^{2t} + 7c_2 e^{-3t} \end{pmatrix}$$

The slides below slides from 250a, fall semester.

You can use them to review eigenvalues & eigenvectors.

You can find all slides from 250a, fall semester at

<http://math.arizona.edu/~cushing/250apresentations.html>

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Motivated by the solution formula $x_n = r^n x_0$ for one recursion equation, let's look for exponential solutions.

Question: Can we find constants r and c_1, c_2 so that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \quad \text{is a solution ?}$$

i.e., so that
$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \quad \text{for all } n = 0, 1, 2, 3, \dots$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2
must satisfy

$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

for all n .

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2
must satisfy

$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} c_1 r^n + a_{12} c_2 r^n \\ a_{21} c_1 r^n + a_{22} c_2 r^n \end{pmatrix}$$

for all n .

$$c_1 r^{n+1} = a_{11} c_1 r^n + a_{12} c_2 r^n$$

$$c_2 r^{n+1} = a_{21} c_1 r^n + a_{22} c_2 r^n$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Cancel r^n

$$c_1 r = a_{11} c_1 + a_{12} c_2$$

$$c_2 r = a_{21} c_1 + a_{22} c_2$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2
must satisfy

$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11}c_1 r^n + a_{12}c_2 r^n \\ a_{21}c_1 r^n + a_{22}c_2 r^n \end{pmatrix} \quad \text{for all } n .$$

Cancel r^n

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

and gather terms

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Two equations, two unknowns c_1, c_2 .

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2
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$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} c_1 r^n + a_{12} c_2 r^n \\ a_{21} c_1 r^n + a_{22} c_2 r^n \end{pmatrix} \quad \text{for all } n .$$

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Equations have only the solution $c_1 = 0, c_2 = 0$,
unless the determinant equals 0 :

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2 must satisfy

$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11}c_1 r^n + a_{12}c_2 r^n \\ a_{21}c_1 r^n + a_{22}c_2 r^n \end{pmatrix} \quad \text{for all } n .$$

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Equations have only the solution $c_1 = 0, c_2 = 0$,
unless the determinant equals 0 :

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

Pick r as a root of this equation --- then solve for c_1, c_2

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \text{ is a solution if } r \text{ satisfies the quadratic equation}$$

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \text{ is a solution if } r \text{ satisfies the quadratic equation}$$

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

A quadratic equation (the *characteristic equation*)

Eigenvalue / Eigenvector Method

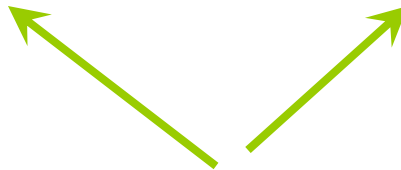
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$ is a solution if r satisfies the quadratic equation

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$



Recall: we need solve only one of these two equations.

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \text{ is a solution if } r \text{ satisfies the quadratic equation}$$

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

r is called an *eigenvalue* of the matrix

They build solutions $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called an *eigenvector* associated with r

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

c = any number

Eigenvalue $r = -0.8$

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

c = any number

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$0.8c_2 + 1.2c_2 = 0$$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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c = any number

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

~~$$0.8c_2 + 1.2c_2 = 0$$~~

Need solve only one equation.

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1.5c \\ c \end{pmatrix}$$

$c = \text{any real number}$

Population Dynamic Application

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1.5c \\ c \end{pmatrix}$$

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Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} -1.5 \times (-0.8)^n \\ (-0.8)^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} -1.5 \times (-1)^n \times 0.8^n \\ (-1)^n \times 0.8^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n c \begin{pmatrix} -1.5 \times 0.8^n \\ 0.8^n \end{pmatrix}$$

$c = \text{any number}$