

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 8-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

$f(x, y) = 0$ defines a curve called the *vertical nullcline*

$g(x, y) = 0$ defines a curve called the *horizontal nullcline*

Nullclines divide phase plane into regions

in which

all vector field arrows point in the same “compass direction”

Example

Pesticide example

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

$-2x + 2y = 0 \Rightarrow$ vertical nullcline is a straight line $y = x$

$2x - 5y = 0 \Rightarrow$ horizontal nullcline is a straight line $y = (2 / 5)x$

Example

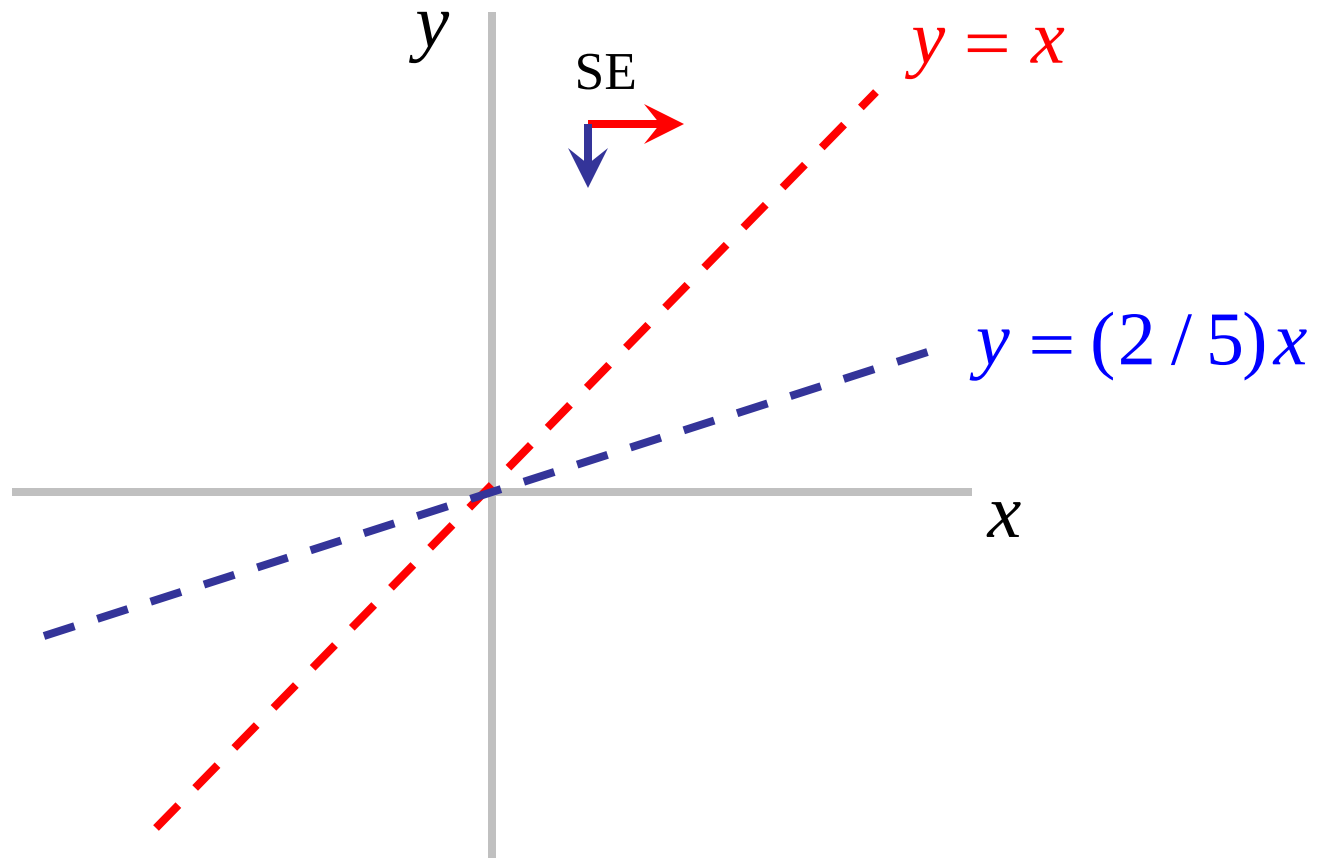
Pesticide example

$$x' = -2x + 2y$$

vertical nullcline is $y = x$

$$y' = 2x - 5y$$

horizontal nullcline $y = (2/5)x$



Example

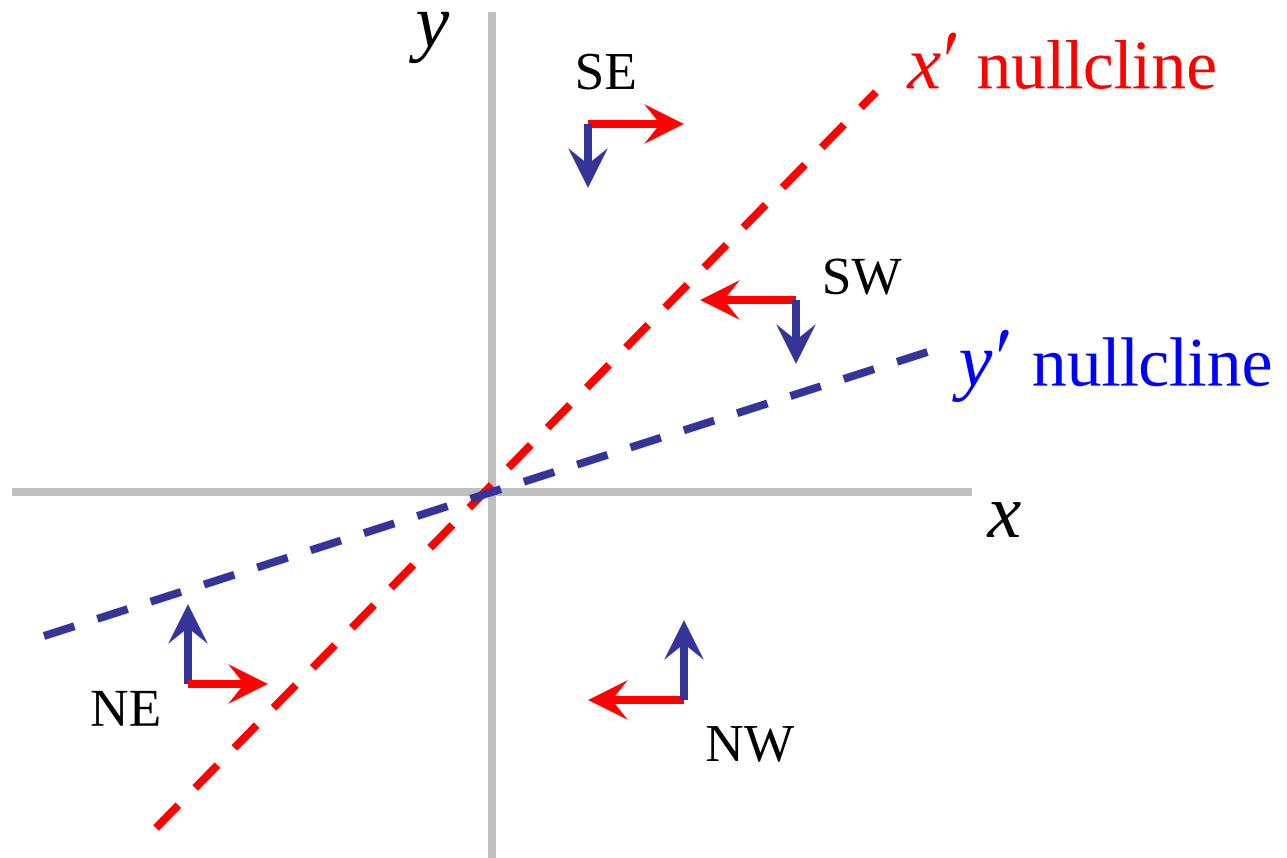
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Example

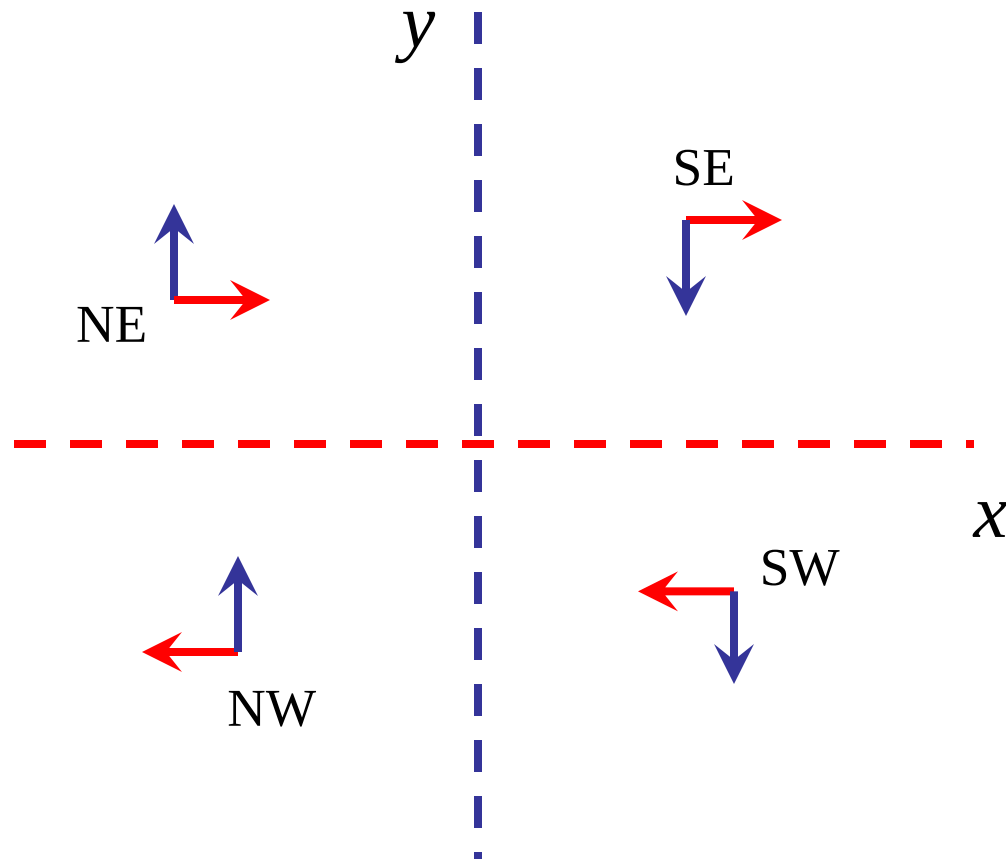
Frictionless Mass-Spring (harmonic oscillator)

$$x' = y$$

vertical nullcline is $y = 0$ (x -axis)

$$y' = -x$$

horizontal nullcline $x = 0$ (y -axis)



LINEAR SYSTEMS

$$x' = a(t)x + b(t)y + h_1(t)$$

$$y' = c(t)x + d(t)y + h_2(t)$$

$$x(t_0) = x_0$$

$$y(t_0) = y_0$$

$$a(t), b(t), c(t), d(t), h_1(t), h_2(t)$$

continuous on an interval $\alpha < t < \beta$

Fundamental Existence & Uniqueness Theorem applies.

A stronger Theorem for linear systems implies
solutions exist on the whole interval $\alpha < t < \beta$.

EXAMPLES

$$x' = ax + by$$

$$y' = cx + dy \quad a, b, c, d \text{ are constants}$$

$$x(t_0) = x_0 \quad \text{an } \textit{autonomous linear} \text{ system}$$

$$y(t_0) = y_0$$

Solutions exist *for all* t , i.e., on the interval $-\infty < t < \infty$

EXAMPLES

$$ax'' + bx' + cx = 0$$

b, c are constants

$x(t_0) = x_0, \quad x'(t_0) = y_0$ *autonomous linear* second order equation

Equivalent autonomous linear first order system

$$x' = y$$

$$y' = -(c/a)x - (b/a)y$$

$$x(t_0) = x_0, \quad y(t_0) = y_0$$

Solutions exist *for all* t , i.e., on the interval $-\infty < t < \infty$

Homogeneous Linear Systems

$$x' = a(t)x + b(t)y$$

$$y' = c(t)x + d(t)y$$

Nonhomogeneous Linear Systems

$$x' = a(t)x + b(t)y + h_1(t)$$

$$y' = c(t)x + d(t)y + h_2(t)$$

Theorem 3 (p. 156)

The *general solution* of a nonhomogeneous system has the form

$$x = x_h(t) + x_p(t)$$

$$y = y_h(t) + y_p(t)$$

where $x_p(t), y_p(t)$ is a *particular* solution pair and

$x_h(t), y_h(t)$ is the *general* solution pair of the *homogeneous* system.

In matrix notation ...

Theorem 3 (p. 156)

The *general solution* of a nonhomogeneous system

$$\hat{x}' = A(t)\hat{x} + \hat{h}(t)$$

has the form

$$\hat{x} = \hat{x}_h(t) + \hat{x}_p(t)$$

where $\hat{x}_h(t) =$ *general solution* of $\hat{x}' = A(t)\hat{x}$

$\hat{x}_p(t) =$ *a particular solution* $\hat{x}' = A(t)\hat{x} + \hat{h}(t)$.

$$\text{RECALL: } \hat{x} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \hat{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$$

Homogeneous Linear Systems

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The general solution has the form

$$\begin{pmatrix} x_h \\ y_h \end{pmatrix} = c_1 \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + c_2 \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$$

where $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}$ and $\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ are two *independent* solution pairs



means

“are not multiples of each other”

Homogeneous Linear Systems

$$\hat{x}' = A(t)\hat{x}$$

The general solution has the form

$$\hat{x} = c_1 \hat{x}_1 + c_2 \hat{x}_2$$

where $\hat{x}_1$ and $\hat{x}_2$ are two *independent* solutions

called a

“a *linear combination*”

means

“*are not multiples of each other*”

Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix}$$

$$\hat{x}_1 = \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix} \quad \text{Not multiples of each other}$$

General Solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2e^{-t} \\ e^{-t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-6t} \\ -2e^{-6t} \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2c_1e^{-t} + c_2e^{-6t} \\ c_1e^{-t} - 2c_2e^{-6t} \end{pmatrix}$$

Examples

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

$$x = 2c_1e^{-t} + c_2e^{-6t}$$

$$y = c_1e^{-t} - 2c_2e^{-6t}$$

Examples

$$\hat{x}' = A(t)\hat{x} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{x}_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \hat{x}_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

Not multiples
of each other

General
Solution

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

or

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t \\ -c_1 \sin t + c_2 \cos t \end{pmatrix}$$

Examples

$$x' = y$$

$$y' = -x$$

$$x = c_1 \cos t + c_2 \sin t$$

$$y = -c_1 \sin t + c_2 \cos t$$

$$x'' = -x$$

$$x = c_1 \cos t + c_2 \sin t$$