

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 1-Apr-08

<http://math.arizona.edu/~cushing/250b.html>

## Examples

(pesticide application)

$$\begin{aligned}x' &= r_2 y - r_1 x & x(0) &= x_0 \\y' &= r_1 x - r_2 y - r_3 y & y(0) &= 0\end{aligned}$$

At time  $t = 0$  an application of  $x_0$  units of pesticide is sprayed on the trees, but there is no pesticide initially in the soil.

Some questions:

- Does the pesticide disappear from the system over time ?
- What is the maximum concentration that accumulates in the soil and when does it occur?
- What fraction of the pesticide in the system resides in the soil?

## Examples

(mass-spring)

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{c}{m}y$$

$$x(0) = x_0$$

$$y(0) = v_0$$

At time  $t = 0$  the mass is displaced  $x_0$  units and given an initial velocity  $v_0$ .

Some questions:

- Does the mass oscillate (around its rest point) ? If so, what's the period, amplitude, etc. ?
- What happens in the long run ?
- How do the answers depend on the mass  $m$  ? The spring constant  $k$  ?  
The coefficient of friction  $c$  ?

A general *first order system* has the form :

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

*Initial value problems* have the form

$$x(t_0) = x_0$$

$$y(t_0) = y_0$$

$$\begin{aligned}x' &= f(t, x, y) & x(t_0) &= x_0 \\y' &= g(t, x, y) & y(t_0) &= y_0\end{aligned}$$

When can we be sure a solution exists ? Is unique ?

### Fundamental Existence & Uniqueness Theorem

Assume  $f(t, x, y), g(t, x, y)$

$$\frac{df(t, x, y)}{dx}, \quad \frac{df(t, x, y)}{dy}, \quad \frac{dg(t, x, y)}{dx}, \quad \frac{dg(t, x, y)}{dy}$$

are all continuous near  $t = t_0, x = x_0, y = y_0$ .

Then there is a unique solution pair  $x(t), y(t)$

on an interval containing  $t_0$ .

## Examples

$$x' = r_2 y - r_1 x$$

$$x(0) = x_0$$

$$y' = r_1 x - r_2 y - r_3 y$$

$$y(0) = 0$$

has a unique solution pair (near  $t_0 = 0$ ) for an values of  $r_1, r_2, r_3$ , and  $x_0$  .

## Examples

$$x' = y$$

$$x(0) = x_0$$

$$y' = -\frac{k}{m}x - \frac{c}{m}y$$

$$y(0) = v_0$$

has a unique solution pair (near  $t_0 = 0$ ) for an values of  $m, k, c, x_0$  and  $v_0$ .

or equivalently ...

$$mx'' = -kx - cx'$$

$$x(0) = x_0, x'(0) = v_0$$

has a unique solution  $x(t)$  (near  $t_0 = 0$ ) for an values of  $m, k, c, x_0$  and  $v_0$ .

How best to display the numerical output ?

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

1. Individual “time series” plots of  $x$  *against*  $t$  and  $y$  *against*  $t$  .
2. 3D plot of  $(t, x, y)$
3. For autonomous systems we can use phase plane plots.

$$x' = f(x, y)$$

$$y' = g(x, y)$$

## Example

Pesticide model

$$x' = -r_1 x + r_2 y$$

$$y' = r_1 x - (r_2 + r_3) y$$

$$r_1 = r_2 = 2, r_3 = 3$$

Use computer to obtain plots  
of  $x$  and  $y$  against  $t$

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

$$x(0) = x_0, y(0) = 0$$

## Example

### Mass-spring model

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{c}{m}y$$

$$m = 1, \quad k = 4,$$

$$c = 0 \quad (\text{frictionless})$$

Use computer to obtain plots  
of  $x$  and  $y$  against  $t$

$$x' = y$$

$$y' = -x$$

$$x(0) = 1, \quad y(0) = 0$$

initial displacement

no initial velocity

$$x' = y$$

$$y' = -x$$

$$x(0) = 0, \quad y(0) = -1$$

no initial displacement

initial velocity (upward)

## Example

### Mass-spring model

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{c}{m}y$$

$$m = 1, \quad k = 4,$$

$$c = 0.5 \quad (\text{with friction})$$

Use computer to obtain plots  
of  $x$  and  $y$  against  $t$

$$x' = y$$

$$y' = -x - 0.5y$$

$$x(0) = 1, \quad y(0) = 0$$

initial displacement

no initial velocity

How best to display the numerical output ?

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

1. Individual “time series” plots of  $x$  *against*  $t$  and  $y$  *against*  $t$  .
2. 3D plot of  $(t, x, y)$
3. For autonomous systems we can use phase plane plots.

$$x' = f(x, y)$$

$$y' = g(x, y)$$