

MATH 250b – Section 2

Spring Semester 2008

Thursday, 27-Mar-08

<http://math.arizona.edu/~cushing/250b.html>

Solutions

A solution $x(t)$ of a higher equation on an interval $a < t < b$
reduces the equation to an identity on the interval

A *solution pair* $x(t), y(t)$ of a system on an interval $a < t < b$
reduces *both* equations to identities on the interval

Example
(the harmonic oscillator)

$$x'' + x = 0$$

CLAIM: $x(t) = \sin t$

is a solution on the interval

$$-\infty < t < \infty$$

PROOF

$$\begin{aligned} x''(t) + x(t) &= (\sin t)'' + \sin t \\ &= -\sin^2 t + \sin t \end{aligned}$$

For all t this is identical
to the right side 0 of equation

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For all t this is identical
to the right side 0 of equation

$$x' = y$$

$$y' = -x$$

CLAIM: $x(t) = \sin t, y(t) = \cos t$

is a solution *pair* on the interval

$$-\infty < t < \infty$$

PROOF

$$x'(t) = (\sin t)'$$

is identical to $y(t) = \cos t$

$$y' = (\cos t)'$$

is identical to $-x = -\sin t$

Example

(the pesticide application)

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

$$\text{CLAIM: } x(t) = 2e^{-t}, \quad y(t) = e^{-t}$$

is a *solution pair* for all t

PROOF

$$\left. \begin{aligned} x'(t) &= (2e^{-t})' = -2e^{-t} \\ -2x(t) + 2y(t) &= -4e^{-t} + 2e^{-t} = -2e^{-t} \end{aligned} \right\} \begin{array}{l} \text{first} \\ \text{equation} \end{array}$$

$$\left. \begin{aligned} y'(t) &= (e^{-t})' = -e^{-t} \\ 2x(t) - 5y(t) &= 4e^{-t} - 5e^{-t} = -e^{-t} \end{aligned} \right\} \begin{array}{l} \text{second} \\ \text{equation} \end{array}$$

Linear Equations

- A higher order equation is *linear* if the unknown (state variable) and all of its derivatives appear linearly
- A system is *linear* if all unknowns (state variables) and all their derivatives appear linearly

Examples

Harmonic Oscillator

$$x'' + x = 0 \quad \text{Linear, second order equation}$$

Pesticide Application

$$\begin{aligned} x' &= -2x + 2y \\ y' &= 2x - 5y \end{aligned} \quad \text{Linear, first order system}$$

Linear Equations

- A higher order equation is *linear* if the unknown (state variable) and all of its derivatives appear linearly
- A system is *linear* if all unknowns (state variables) and all their derivatives appear linearly

Examples

Periodically Forced Harmonic Oscillator

$$x'' + x = \cos t \quad \text{Linear, second order equation}$$

Continuous Pesticide Application to the Trees

$$\begin{aligned} x' &= -2x + 2y + 10 \\ y' &= 2x - 5y \end{aligned} \quad \text{Linear, first order system}$$

Linear Equations

- A higher order equation is *linear* if the unknown (state variable) and all of its derivatives appear linearly
- A system is *linear* if all unknowns (state variables) and all their derivatives appear linearly

Examples

van der Pol Equation

$$x'' + a(x^2 - 1)x' + x = 0 \quad \text{Nonlinear, second order equation}$$

Volterra's Predator-Prey System

$$\begin{aligned} x' &= rx \left(1 - \frac{x}{K} \right) - xy \\ y' &= -ry + cxy \end{aligned} \quad \text{Nonlinear, first order system}$$

Matrix Notation & Algebra

Addition and scalar multiplication are done component-wise :

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$$

$$c \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ca_1 \\ ca_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

$$c \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$$

Matrix Notation & Algebra

Matrix Multiplication :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

Matrix Notation & Algebra

Matrix & Vectors of Functions :

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

$$\int \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} dt = \begin{pmatrix} \int x(t) dt \\ \int y(t) dt \end{pmatrix}$$

Examples

$$\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}' = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\int \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} dt = \begin{pmatrix} -\cos t + c_1 \\ \sin t + c_2 \end{pmatrix} = \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Writing Systems in Matrix Notation

Example

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

Write unknown pair as a vector :

$$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2x + 2y \\ 2x - 5y \end{pmatrix}$$

Writing Systems in Matrix Notation

Example

$$x' = -2x + 2y$$

$$y' = 2x - 5y$$

For a linear system like this, we can make more use of matrix notation :

$$A = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} = \text{the coefficient matrix}$$

$$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \text{the solution vector}$$

We can write the system as $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ or $\bar{x}' = A\bar{x}$

Writing Systems in Matrix Notation

Example

$$x'' + x = 0 \quad \text{or} \quad \begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \bar{x}' = A\bar{x}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \text{the coefficient matrix}$$

Most general linear system has the form :

$$x' = a(t)x + b(t)y + h_1(t)$$

$$y' = c(t)x + d(t)y + h_2(t)$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$$

$$A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} = \text{the coefficient matrix}$$

$$\bar{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \text{the nonhomogeneous (forcing) term}$$

Most general linear system has the form :

$$\bar{x}' = A\bar{x} + \bar{h}(t)$$

where ...

$$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \text{the } \textit{solution vector}$$

$$A = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} = \text{the } \textit{coefficient matrix}$$

$$\bar{h}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} = \text{the } \textit{nonhomogeneous (forcing) term}$$