

MATH 250b – Section 2

Spring Semester 2008

Thursday, 06-Mar-08

<http://math.arizona.edu/~cushing/250b.html>

Chapter 3.2: Separable Equations

Goal : find *formulas for solutions*

Methods are sometimes available for *special types* of equations

Example: Method of Separation of Variables is available for

Separable Equations

(autonomous equations are separable equations)

Autonomous Equations

$$x' = f(x)$$

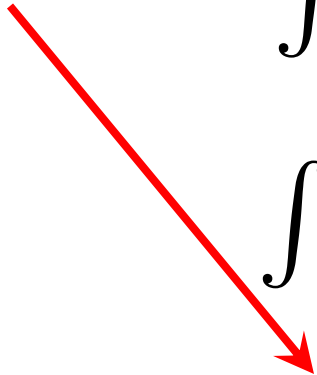
$$\frac{dx}{dt} = f(x)$$

Ignore
equilibria


$$\frac{1}{f(x)} dx = dt$$

(separate variables)

... until the
end


$$\int \frac{1}{f(x)} dx = \int dt$$

(anti-differentiate)

$$\int \frac{1}{f(x)} dx = t + k$$

General Solution

(Implicit Form)

plus equilibria

Example

$$x' = (1 - x)x$$

$$\frac{dx}{dt} = (1 - x)x$$

$$\frac{1}{(1 - x)x} dx = dt$$

(separate variables)

$$\int \frac{1}{(1 - x)x} dx = \int dt$$

(anti-differentiate)

$$\ln \left| \frac{x}{x - 1} \right| = t + k$$

General Solution

Implicit Form

plus equilibria $x = 0, 1$

k = arbitrary constant

Example

$$x' = (1 - x)x$$

$$\frac{dx}{dt} = (1 - x)x$$

$$\frac{1}{(1 - x)x} dx = dt$$

(separate variables)

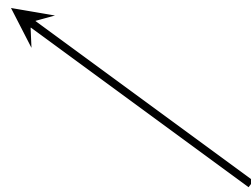
$$\int \frac{1}{(1 - x)x} dx = \int dt$$

(anti-differentiate)

$$x = \frac{1}{1 - ce^{-t}}, \quad c = \pm e^k$$

plus equilibria $x = 0, 1$

arbitrary
nonzero constant



Example

$$x' = (1 - x)x$$

$$\frac{dx}{dt} = (1 - x)x$$

$$\frac{1}{(1 - x)x} dx = dt$$

(separate variables)

$$\int \frac{1}{(1 - x)x} dx = \int dt$$

(anti-differentiate)

$$x = \frac{1}{1 - ce^{-t}}$$

General Solution

plus equilibrium $x = 0$

Explicit Form

c = arbitrary constant

Non-autonomous Separable Equations

$$x' = g(t)h(x)$$

$$\frac{dx}{dt} = g(t)h(x)$$

$$\frac{1}{h(x)} dx = g(t) dt \quad \text{(separate variables)}$$

$$\int \frac{1}{h(x)} dx = \int g(t) dt + k$$

plus equilibria

General Solution
Implicit Form

Example

$$x' = e^{-t} x^2$$

$$\frac{dx}{dt} = e^{-t} x^2$$

$$\frac{1}{x^2} dx = e^{-t} dt \quad (\text{separate variables})$$

$$\int \frac{1}{x^2} dx = \int e^{-t} dt + k$$

$$-\frac{1}{x} = -e^{-t} + k \quad \text{General Solution}$$

plus equilibrium $x = 0$

Implicit Form

$k = \text{arbitrary constant}$

Example: Logistic

$$x' = rx \left(1 - \frac{x}{K} \right)$$

$$\int \frac{1}{x(1 - x/K)} dx = \int r dt + k$$

$$\ln \left| \frac{x}{1 - x/K} \right| = rt + k$$

$$\frac{x}{1 - x/K} = ce^{rt}, \quad c = \pm e^k$$

$$x = \frac{Kce^{rt}}{K + ce^{rt}}$$

plus equilibria 0 and K

Example

Logistic with periodic growth rate r

$$x' = r(1 + \cos t)x \left(1 - \frac{x}{K}\right)$$

$$\int \frac{1}{x(1 - x/K)} dx = \int r(1 + \cos t) dt + k$$

$$\ln \left| \frac{x}{1 - x/K} \right| = r(t + \sin t) + k$$

$$\frac{x}{1 - x/K} = ce^{r(t + \sin t)}, \quad c = \pm e^k$$

$$x = \frac{Kce^{r(1 + \sin t)}}{K + ce^{r(1 + \sin t)}}$$

plus equilibria 0 and K

Initial Value Problems for Separable Equations

$$x' = g(t)h(x), \quad x(t_0) = x_0$$

If $h(x_0) = 0$, then the solution is an equilibrium.

If $h(x_0) \neq 0$, then we separate variables:

$$\frac{1}{h(x)} dx = g(t) dt$$

$$\int_{x_0}^x \frac{1}{h(x)} dx = \int_{t_0}^t g(t) dt$$

Use definite integrals and the Fundamental Theorem of Calculus

Example

$$x' = e^{-t} x^2, \quad x(0) = 2$$

$$\frac{1}{x^2} dx = e^{-t} dt \quad (\text{separate variables})$$

$$\int_2^x \frac{1}{x^2} dx = \int_0^t e^{-t} dt$$

$$-\frac{1}{x} \Big|_2^x = -e^{-t} \Big|_0^t$$

$$-\frac{1}{x} + \frac{1}{2} = -e^{-t} + 1$$

$$x = \frac{2}{2e^{-t} - 1} \text{ on } -\infty < t < \ln 2$$