

MATH 250b – Section 2

Spring Semester 2008

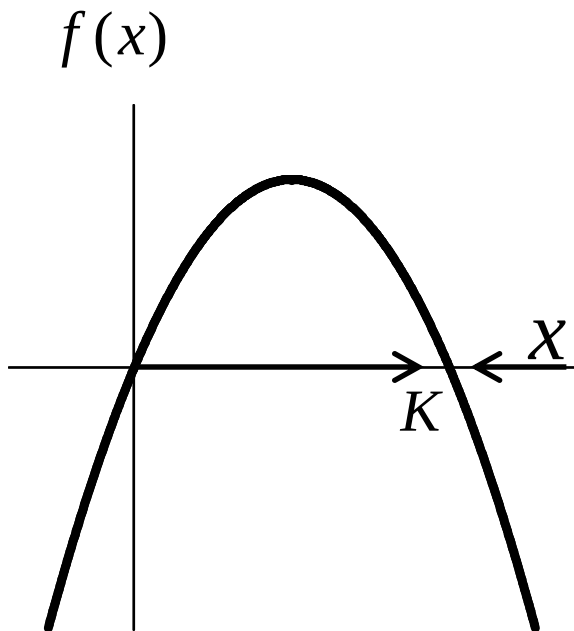
Thursday, 28-Feb-08

<http://math.arizona.edu/~cushing/250b.html>

Phase Portraits Depend on Equation's Parameter Values

Example: Logistic Equation

$$x' = r x \left(1 - \frac{x}{K} \right), \quad r \text{ and } K > 0 \qquad f(x) = r x \left(1 - \frac{x}{K} \right)$$



Phase Portraits Depend on Equation's Parameter Values

Example: Harvested Logistic Equation

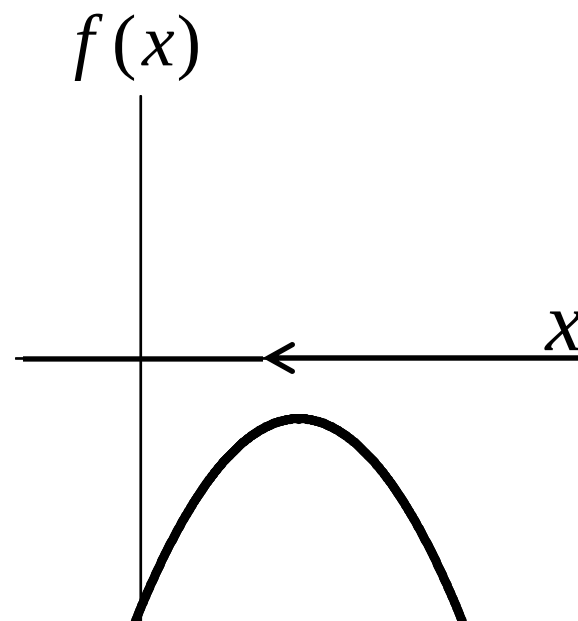
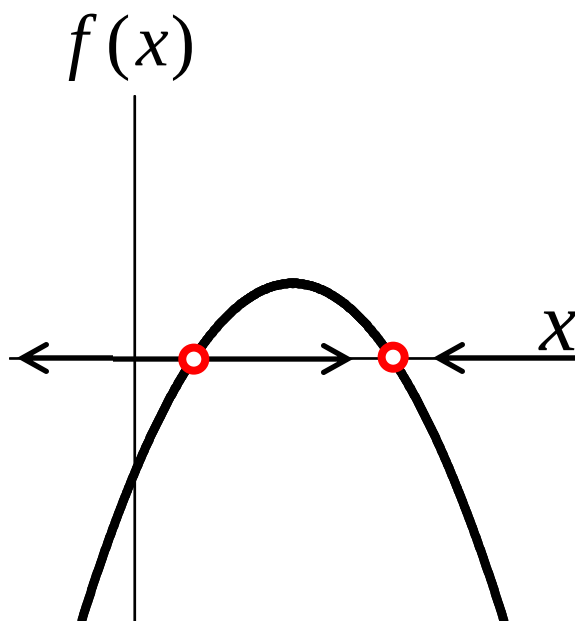
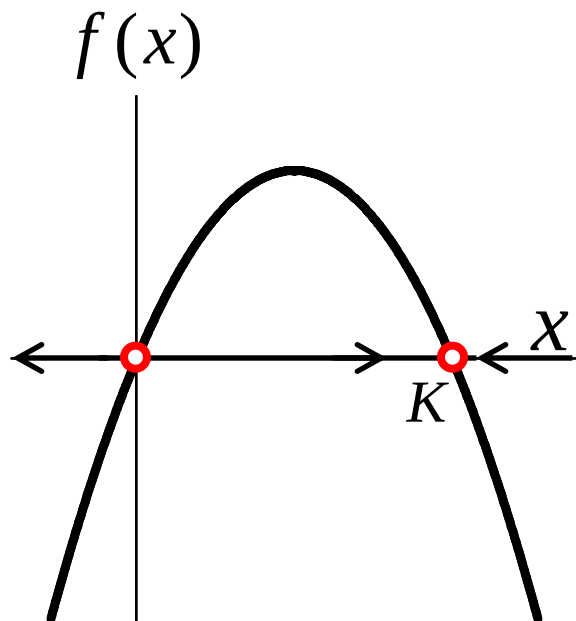
$$x' = r x \left(1 - \frac{x}{K} \right) - h, \quad r, K \text{ and } h > 0$$

$$f(x) = r x \left(1 - \frac{x}{K} \right)$$

$h = 0$

h small

h large



Phase Portraits Depend on Equation's Parameter Values

Example: Harvested Logistic Equation

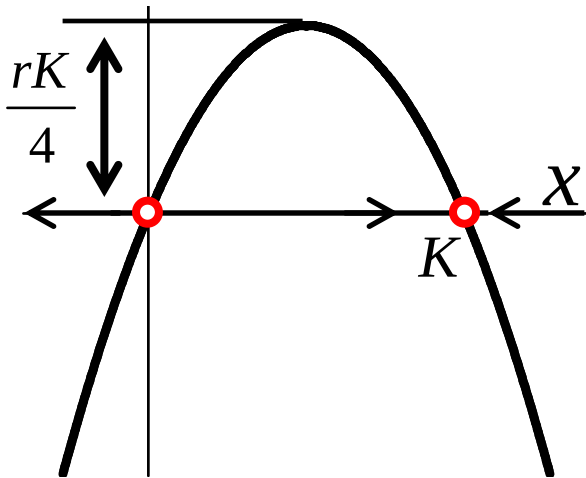
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$$f(x) = r x \left(1 - \frac{x}{K} \right)$$

$$h = 0$$

Sustainable

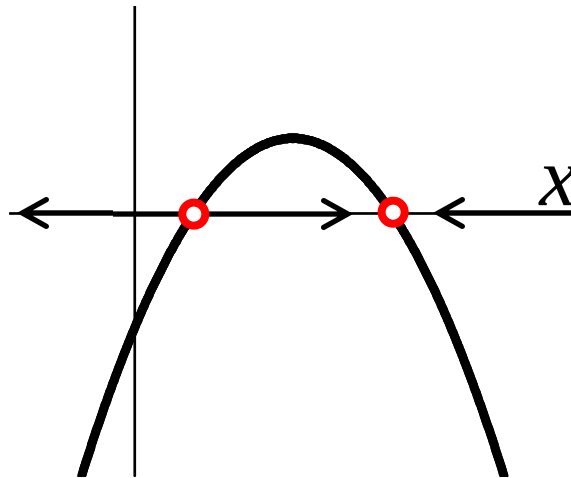
All populations survive



$$0 < h < \frac{rK}{4}$$

Sustainable

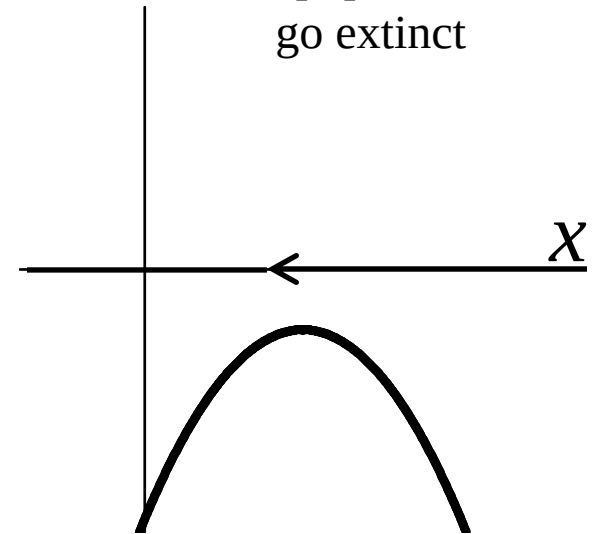
Large populations survive



$$\frac{rK}{4} < h$$

Not sustainable

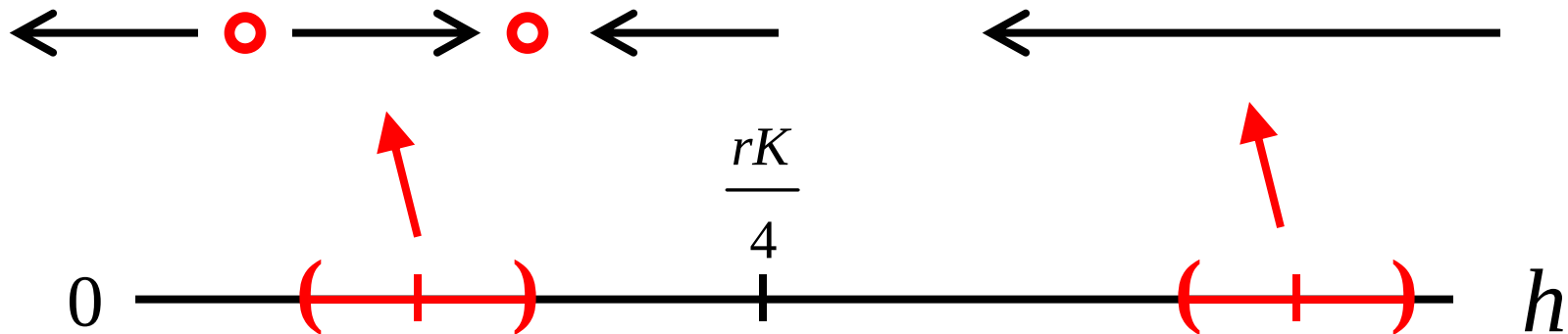
All populations go extinct



$$\frac{rK}{4} = \text{maximum sustainable yield}$$

Bifurcations

$$x' = r x \left(1 - \frac{x}{K} \right) - h$$



Phase portraits remain *qualitatively equivalent* throughout these intervals

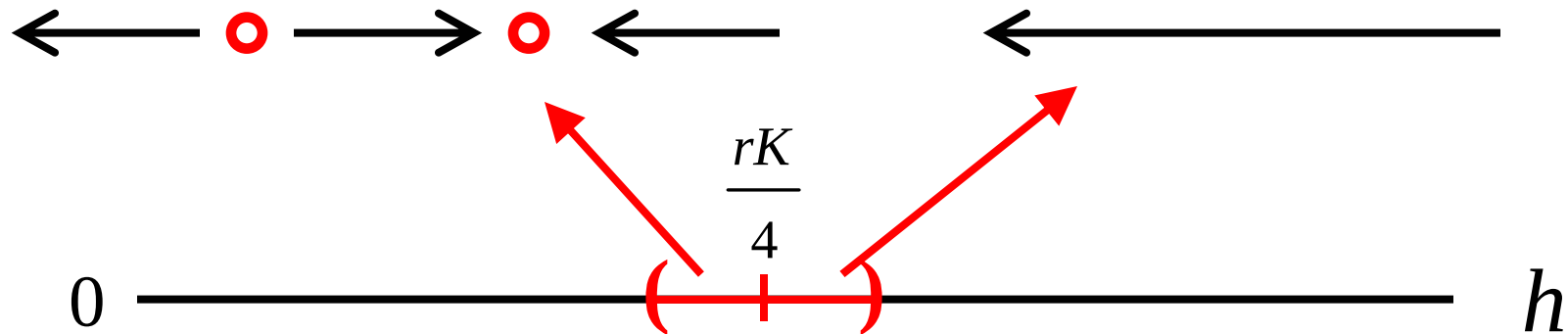
These points are *stable values* for the parameter h .

Qualitatively equivalent means:

1. same number of equilibria
2. same sequence of orbit orientations

Bifurcations

$$x' = r x \left(1 - \frac{x}{K} \right) - h$$



Phase portraits *do not* remain qualitatively equivalent throughout *any* interval.

This point is a *bifurcation value* of the parameter h .

Qualitatively equivalent means:

1. same number of equilibria
2. same sequence of orbit orientations

Bifurcations

$$x' = r x \left(1 - \frac{x}{K} \right) - h$$

- ✓ Using h as a *bifurcation parameter*, this equation undergoes a *bifurcation* at the *bifurcation* (or *critical*) value $h_0 = rK/4$.
- ✓ As h increases through the bifurcation value $h_0 = rK/4$, the number of equilibria changes from 2 to 0.
- ✓ This bifurcation type is called a *saddle-node* (or *blue-sky*) bifurcation. It is characterized by a change in the number of equilibria from 0 to 2 (or 2 to 0).

Example

$$x' = x^2 - p$$

p is a constant (parameter)

$$\text{Equilibrium Equation : } x^2 - p = 0$$

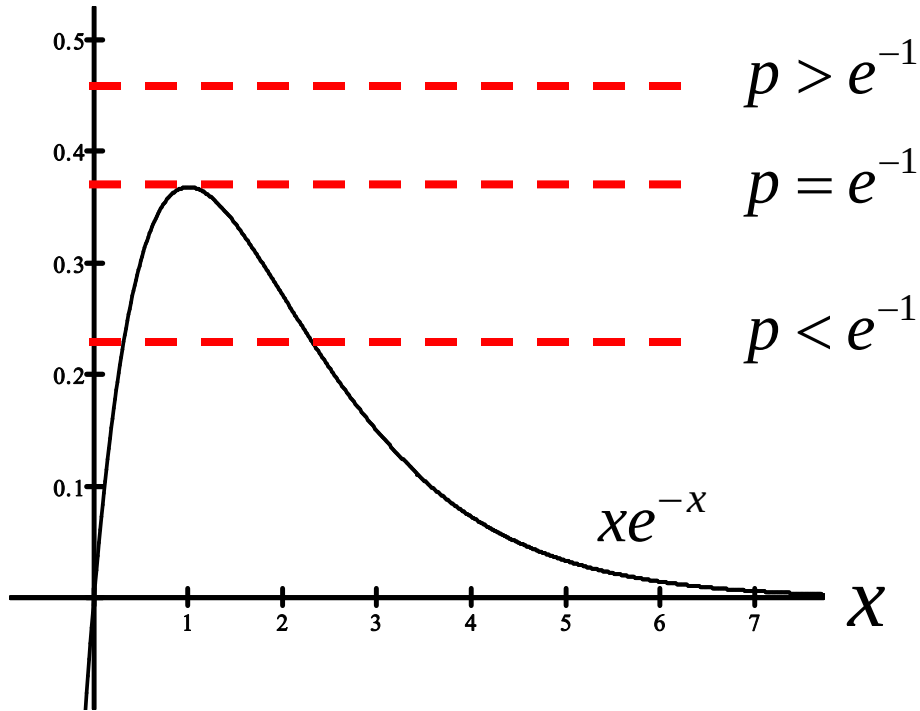
$$\text{Equilibria : } x_e = \begin{cases} \text{none} & \text{if } p < 0 \\ 0 & \text{if } p = 0 \\ \pm\sqrt{p} & \text{if } p > 0 \end{cases}$$

$p = 0$ is a saddle-node bifurcation value

Example

$$x' = xe^{-x} - p, \quad p > 0$$

Equilibrium Equation: $xe^{-x} - p = 0$ or $xe^{-x} = p$



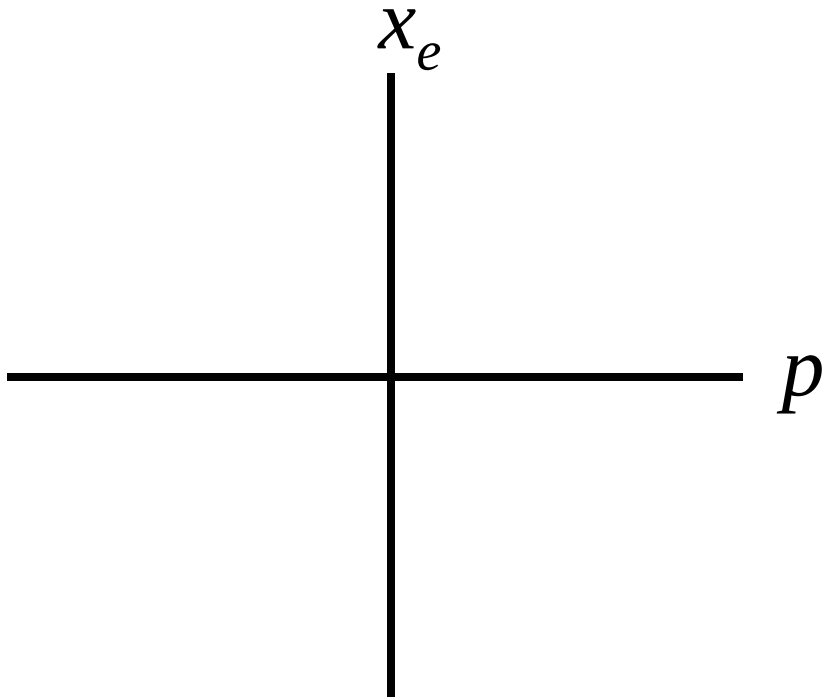
Equilibria :

$$x_e = \begin{cases} \text{none} & \text{if } p > e^{-1} \\ \text{one} & \text{if } p = e^{-1} \\ \text{two} & \text{if } p < e^{-1} \end{cases}$$

$p = e^{-1}$ is a saddle-node bifurcation value

Bifurcation Diagrams

Plot the equilibria x_e against the parameter p in the plane :

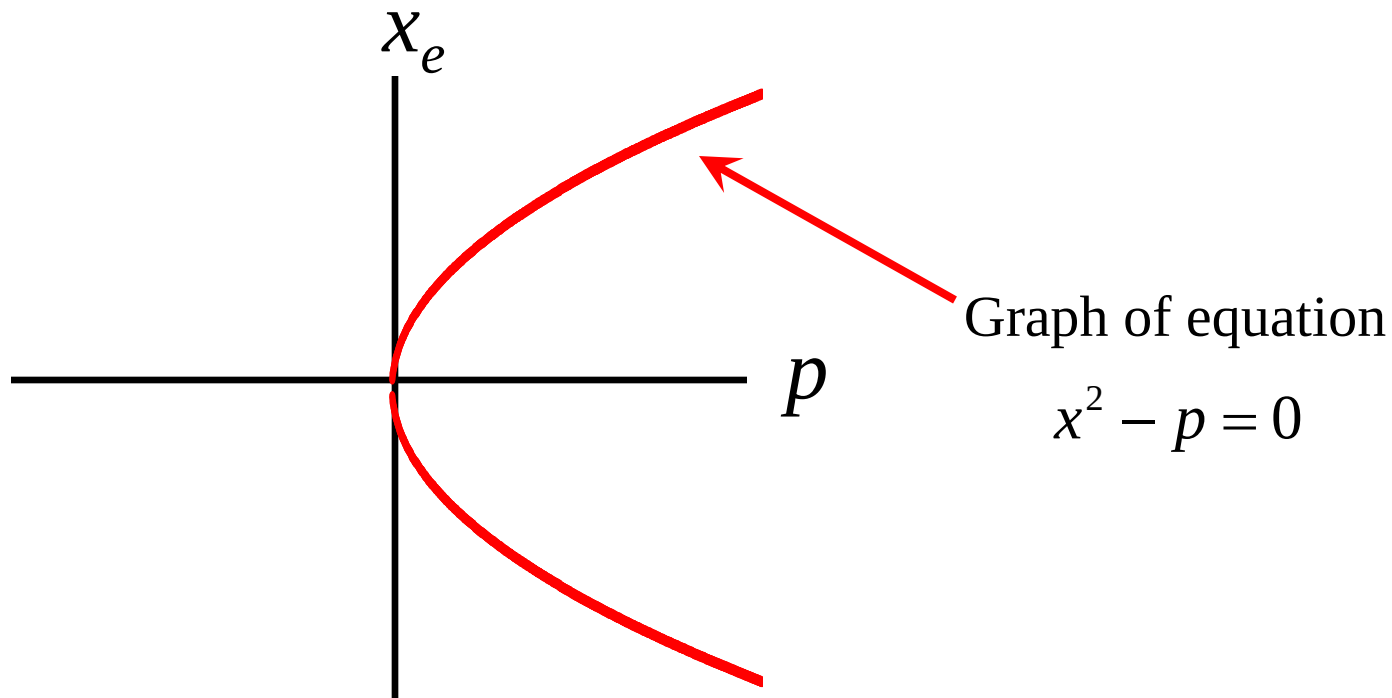


Bifurcation Diagrams

$$x' = x^2 - p$$

p is a constant (parameter)

Equilibrium Equation : $x^2 - p = 0$

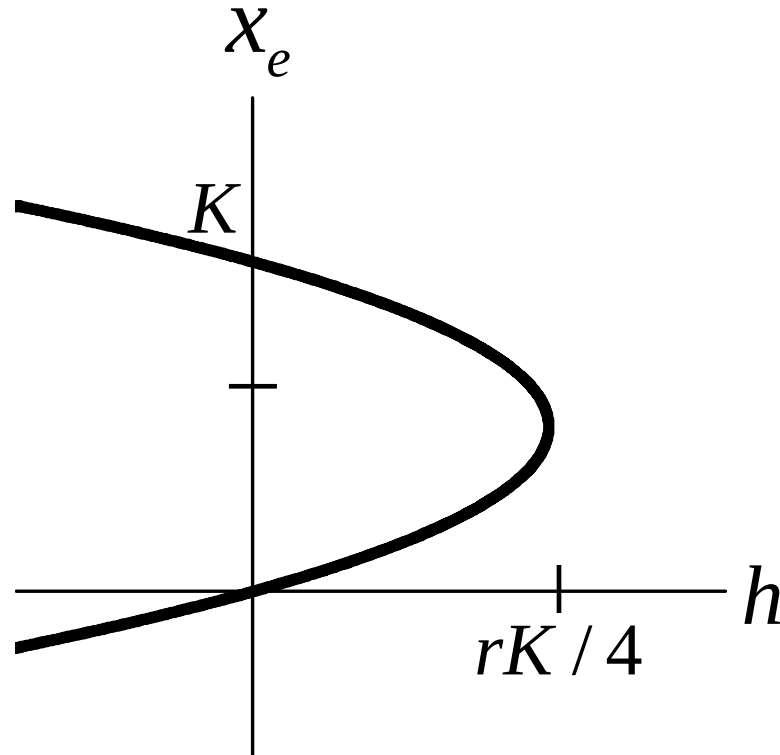


Bifurcation Diagrams

Harvested Logistic Growth

$$x' = rx \left(1 - \frac{x}{K} \right) - h$$

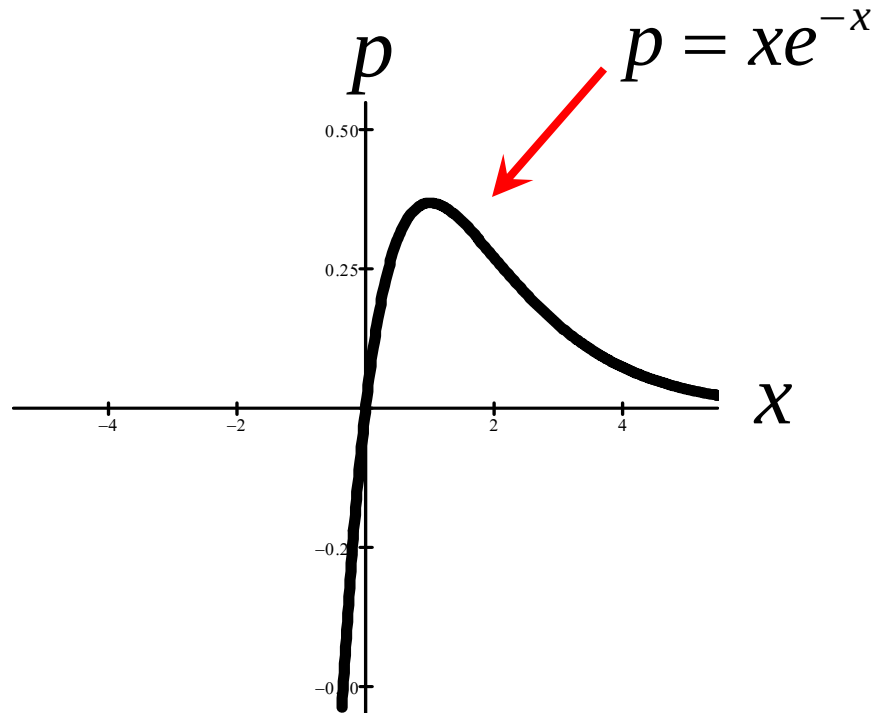
$$r > 0, K > 0$$



Bifurcation Diagrams

$$x' = xe^{-x} - p, \quad p > 0$$

$$xe^{-x} - p = 0$$

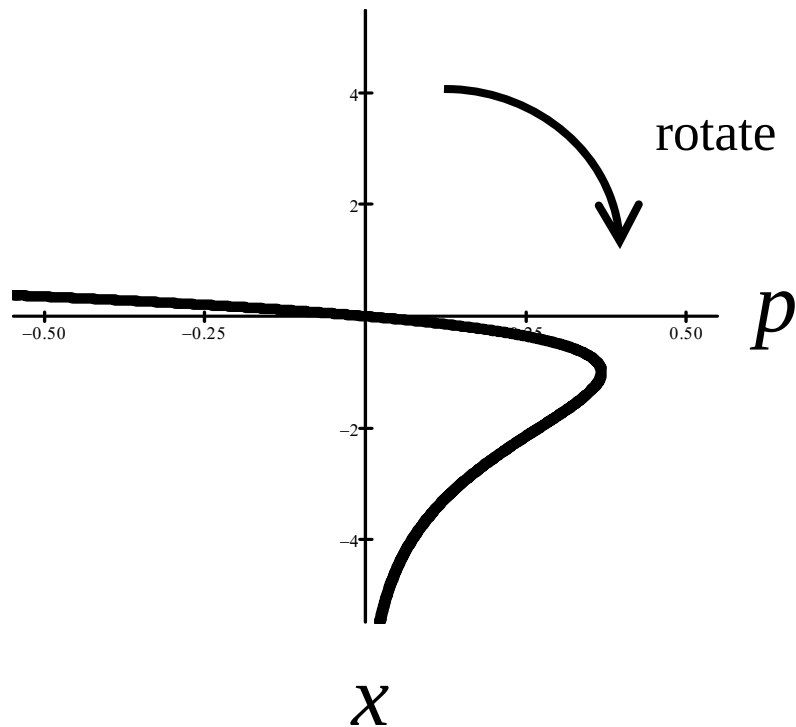


Bifurcation Diagrams

$$x' = xe^{-x} - p, \quad p > 0$$

$$xe^{-x} - p = 0$$

$$p = xe^{-x}$$

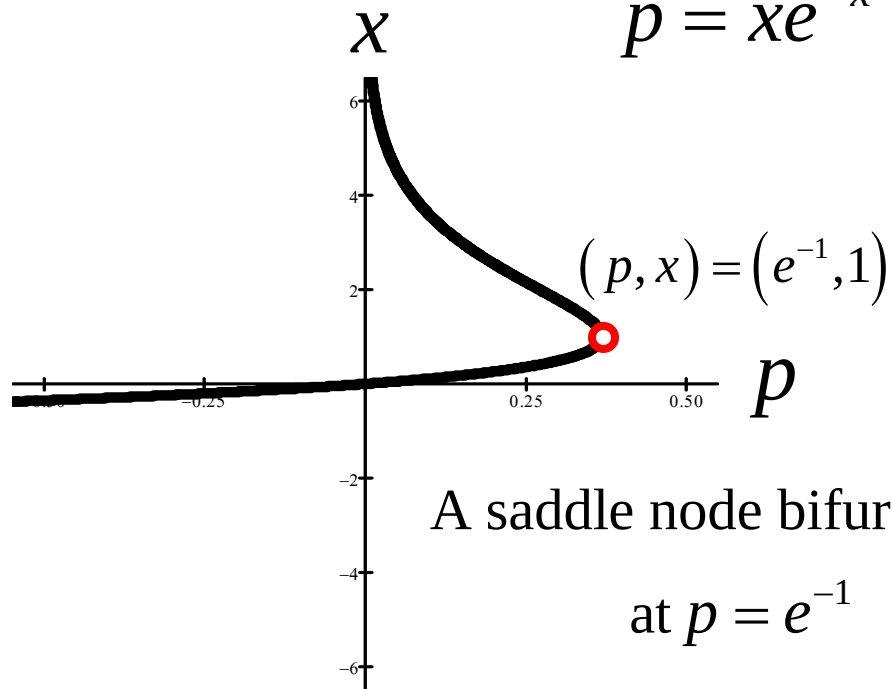


Bifurcation Diagrams

$$x' = xe^{-x} - p, \quad p > 0$$

$$xe^{-x} - p = 0$$

$$p = xe^{-x}$$



So far our bifurcation diagrams depict only the *existence & number* of equilibria.

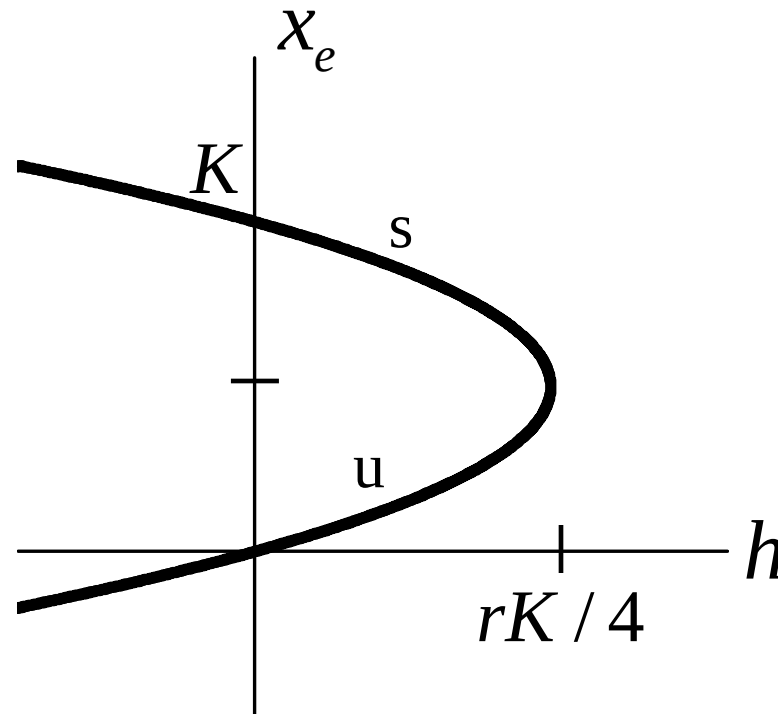
There are several ways to also depict the *stability properties* of equilibria.

Harvested Logistic Growth

$$x' = rx \left(1 - \frac{x}{K} \right) - h$$

$$r > 0, \quad K > 0$$

saddle-node
bifurcation



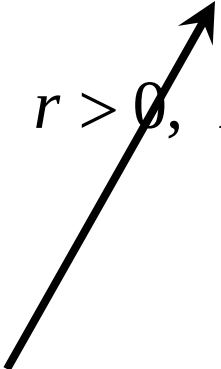
AN APPLICATION

The Spruce Budworm Model

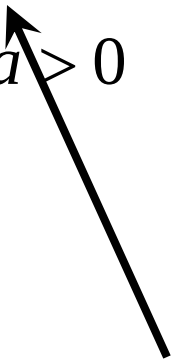
$$x' = rx \left(1 - \frac{x}{K} \right) - c \frac{x^2}{a + x^2}$$

$r > 0, K > 0, c > 0, a > 0$

A logistically
growing population
of budworms



A removal rate
caused by
predators & parasites



AN APPLICATION

The Spruce Budworm Model

$$x' = rx \left(1 - \frac{x}{K} \right) - c \frac{x^2}{a + x^2}$$

$$r > 0, \quad K > 0, \quad c > 0, \quad a > 0$$

An Example

$$r > 0, \quad K = 1, \quad c = 1, \quad a = 0.01$$



A parameter determined by
environmental conditions,
e.g. the “health” of the forest

AN APPLICATION

The Spruce Budworm Model

$$x' = rx(1-x) - \frac{x^2}{0.01 + x^2}$$

Equilibrium Equation:

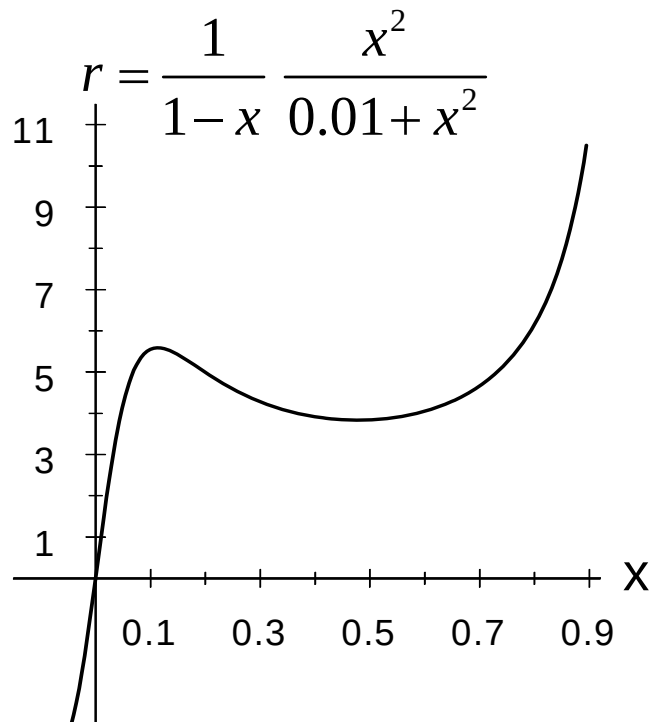
$$rx(1-x) - \frac{x^2}{0.01 + x^2} = 0$$

$$x = 0 \quad \text{and} \quad r = \frac{1}{1-x} \frac{x^2}{0.01 + x^2}$$

AN APPLICATION

The Spruce Budworm Model

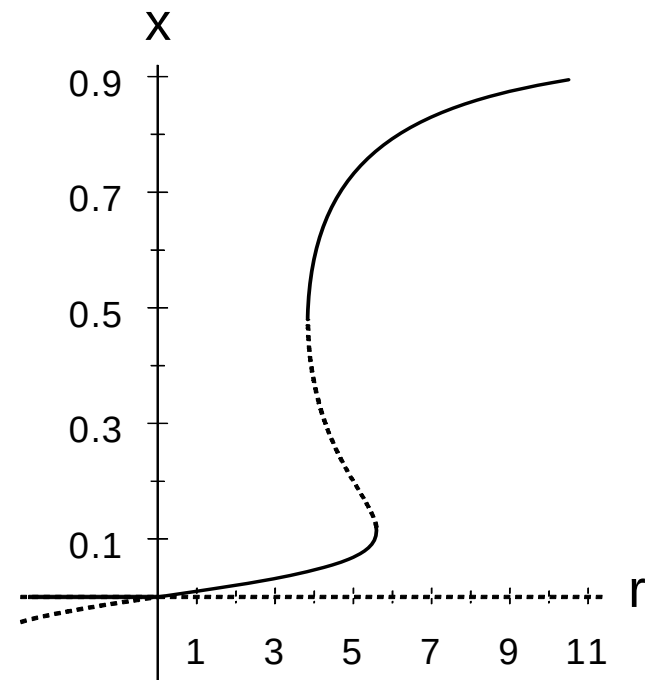
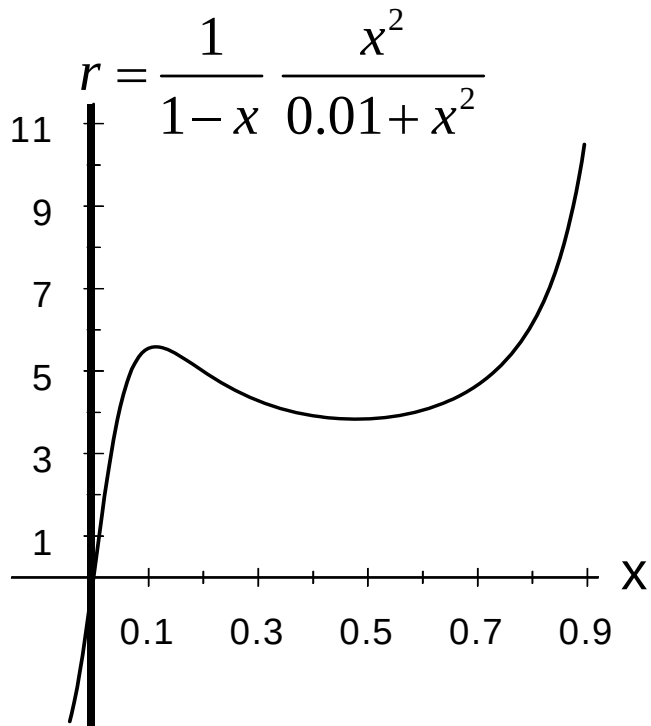
$$x' = rx(1-x) - \frac{x^2}{0.01+x^2}$$



AN APPLICATION

The Spruce Budworm Model

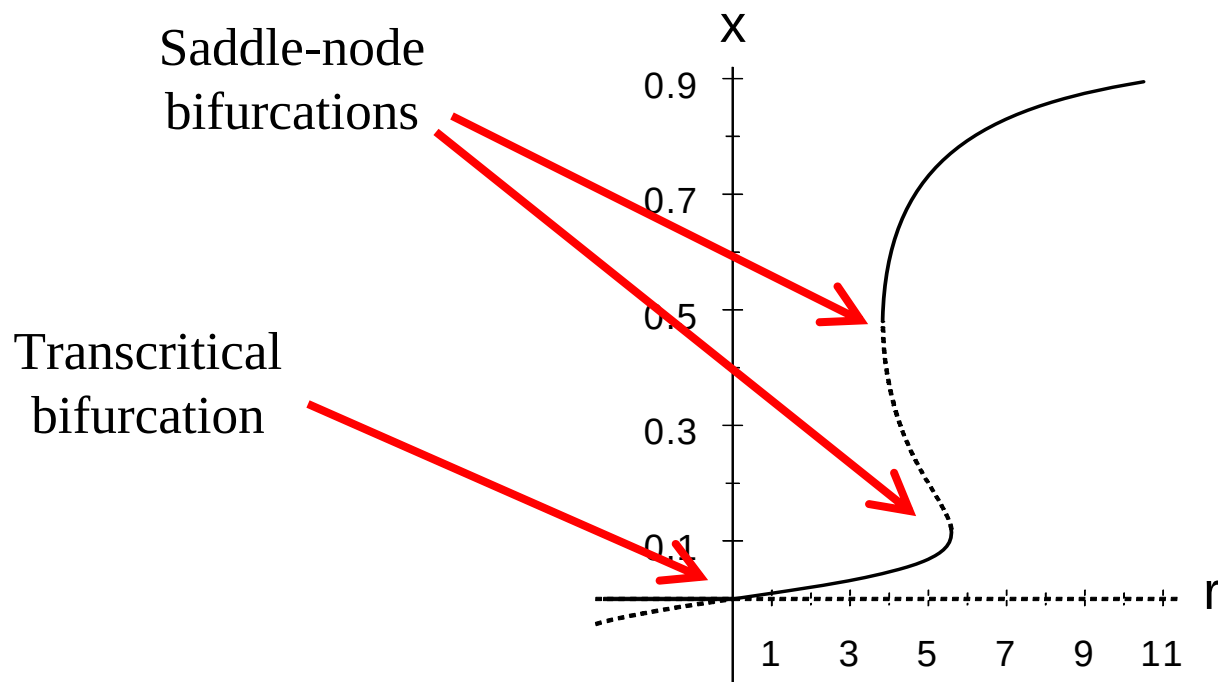
$$x' = rx(1-x) - \frac{x^2}{0.01+x^2}$$



AN APPLICATION

The Spruce Budworm Model

$$x' = rx(1-x) - \frac{x^2}{0.01+x^2}$$



AN APPLICATION

The Spruce Budworm Model

$$x' = rx(1-x) - \frac{x^2}{0.01+x^2}$$

