

MATH 250b – Section 2

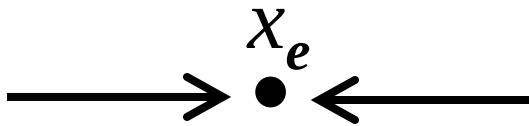
Spring Semester 2008

Tuesday, 26-Feb-08

<http://math.arizona.edu/~cushing/250b.html>

Classification of Equilibria and Geometry of $f(x)$

(Theorem 3 in text)

- ✓ *Attractor* (stable equilibrium) 

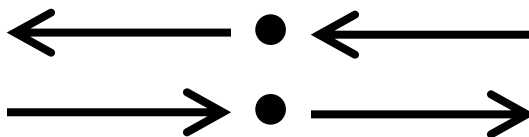
This occurs when the graph of $f(x)$ to *decreases* through the root x_e

$$df / dx \Big|_{x=x_e} < 0 \Rightarrow x_e = \text{attractor}$$

- ✓ *Repeller* (unstable equilibrium) 

This occurs when the graph of $f(x)$ to *increases* through the root x_e

$$df / dx \Big|_{x=x_e} > 0 \Rightarrow x_e = \text{repeller}$$

- ✓ *Shunt* (semi-stable equilibrium) 

This occurs when graph of $f(x)$ has a *max or min* at the root x_e

$$df / dx \Big|_{x=x_e} = 0 \Rightarrow \text{further analysis is needed}$$

SUMMARY

Construction of Phase Line Portraits for $x' = f(x)$

1. Draw a graph of $f(x)$
2. Test points
3. Determine geometry of graph of $f(x)$ at each root

Derivative Test

(Theorem 4)

$df / dx < 0$ at a root $x_e \Rightarrow x_e$ is an *attractor*

$df / dx > 0$ at a root $x_e \Rightarrow x_e$ is a *repeller*

$df / dx = 0$ at a root $x_e \Rightarrow x_e$ "no decision"

$df / dx \Big|_{x=x_e} \neq 0$, then the root (equilibrium) is *hyperbolic*

Classification of Equilibria and Geometry of $f(x)$

- ✓ *Derivative test works for hyperbolic equilibria.*
- ✓ *The Linearization Principle for hyperbolic equilibria :*

If one replaces $f(x)$ by its tangent line at x_e , the resulting linear equation

$$x' = \lambda(x - x_e)$$

and the original equation

$$x' = f(x)$$

have the same type of equilibrium at x_e .

$y' = \lambda y$ is called the *linearization* at x_e

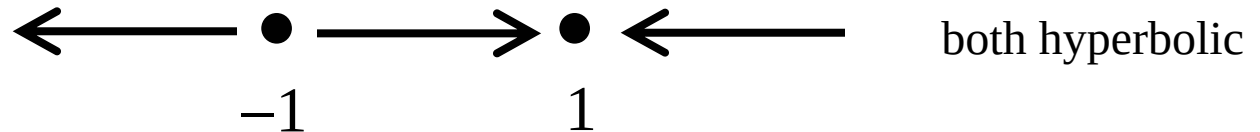
Example

$$x' = 1 - x^2 \Rightarrow f(x) = 1 - x^2$$

$$\text{Equilibrium equation: } 1 - x^2 = 0 \Rightarrow x_e = \pm 1$$

$$\frac{df(x)}{dx} = -2x \Rightarrow \left. \frac{df(x)}{dx} \right|_{x_e} = \begin{cases} -2 & \text{for } x_e = 1 \\ 2 & \text{for } x_e = -1 \end{cases}$$

$$\text{Derivative Test} \Rightarrow \begin{cases} x_e = 1 & \text{is an attractor (stable)} \\ x_e = -1 & \text{is a repeller (unstable)} \end{cases}$$



Linearizations:

At $x_e = 1$: $y' = -2y$ has an attractor

At $x_e = -1$: $y' = 2y$ has a repeller

Example

$$x' = x^2 (1 - x) \Rightarrow f(x) = x^2 (1 - x)$$

Equilibrium equation: $x^2 (1 - x) = 0 \Rightarrow x_e = 0, 1$

$$\frac{df(x)}{dx} = 2x - 3x^2 \Rightarrow \left. \frac{df(x)}{dx} \right|_{x_e} = \begin{cases} 0 & \text{for } x_e = 0 \\ -1 & \text{for } x_e = 1 \end{cases}$$

Derivative Test $\Rightarrow x_e = 1$ is an attractor

$x_e = 0$ is non-hyperbolic and test fails



Linearizations:

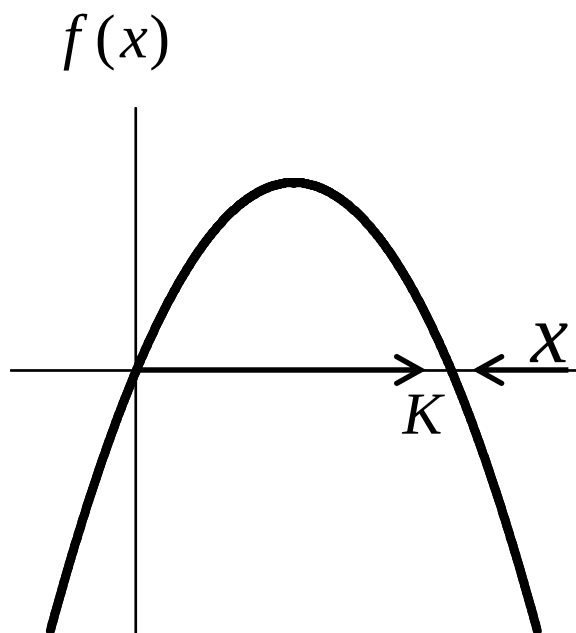
At $x_e = 1$: $y' = -y$ has an attractor

At $x_e = 0$: $y' = 0y$ non-hyperbolic

Phase Portraits Depend on Equation's Parameter Values

Example: Logistic Equation

$$x' = rx \left(1 - \frac{x}{K} \right), \quad r \text{ and } K > 0 \qquad f(x) = rx \left(1 - \frac{x}{K} \right)$$



Phase Portraits Depend on Equation's Parameter Values

Example: Harvested Logistic Equation

$$x' = rx \left(1 - \frac{x}{K} \right) - h, \quad r, K \text{ and } h > 0 \quad f(x) = rx \left(1 - \frac{x}{K} \right) - h$$

$$h = 0$$

$$0 < h < \frac{rK}{4}$$

$$\frac{rK}{4} < h$$

