

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 12-Feb-08

<http://math.arizona.edu/~cushing/250b.html>

Chapter 2: Linear First Order Equations

- What are “linear” equations ?
- Methods for finding solution formulas
- Learn about some basic properties of solutions
- Finish application to falling objects under the Linear Friction Law

2.1 & 2.2 Solution of Linear Equations

Review of Last Class

$$x' = f(t, x)$$

A differential equation is *linear* if $f(t, x)$ is linear in x :

$$x' = p(t)x + q(t)$$

A linear equation is *homogeneous* if $q(t) = 0$:

$$x' = p(t)x$$

The *general solution* of a homogeneous equation is

$$x(t) = ce^{P(t)}, \quad P(t) = \int p(t) dt$$

$c =$ arbitrary constant

2.1 & 2.2 Solution of Linear Equations

$$x' = p(t)x + q(t)$$

When $q(t)$ is present, the equation is *non - homogeneous* :

The *general solution* is

$$x(t) = ce^{P(t)} + e^{P(t)} \int e^{-P(t)} q(t) dt$$

$c =$ arbitrary constant

Structure of General Solution

$$x(t) = x_h(t) + x_p(t)$$

$x_h(t) = ce^{P(t)}$ = general solution of homogeneous equation

$x_p(t) = e^{P(t)} \int e^{-P(t)} q(t) dt =$ a particular solution

Shortcuts

For some equations, one can guess a particular solution $x_p(t)$ and avoid the integration in the general formula.

(Section 2.1.3)

Examples

$x' = -x + 2e^t$ Guess $x_p(t) = ke^t$ and find k .

$x' = x + 1$ Guess $x_p(t) = k$ and find k .

Shortcuts

An Important Example

$x' = px + q$ where p and q are constants.

(equation is called *autonomous*)

Has a constant solution $x_p(t) = -q / p$ (if $p \neq 0$).

A constant solution is called an *equilibrium* (or *steady - state*)

If $p \neq 0$, the equation is called *hyperbolic*.

General solution of hyperbolic equation:

$$x(t) = ce^{pt} - q / p .$$

Linear Autonomous Equations

$$x' = px + q \quad \text{where } p \text{ and } q \text{ are constants}$$

If $p \neq 0$, general solution is

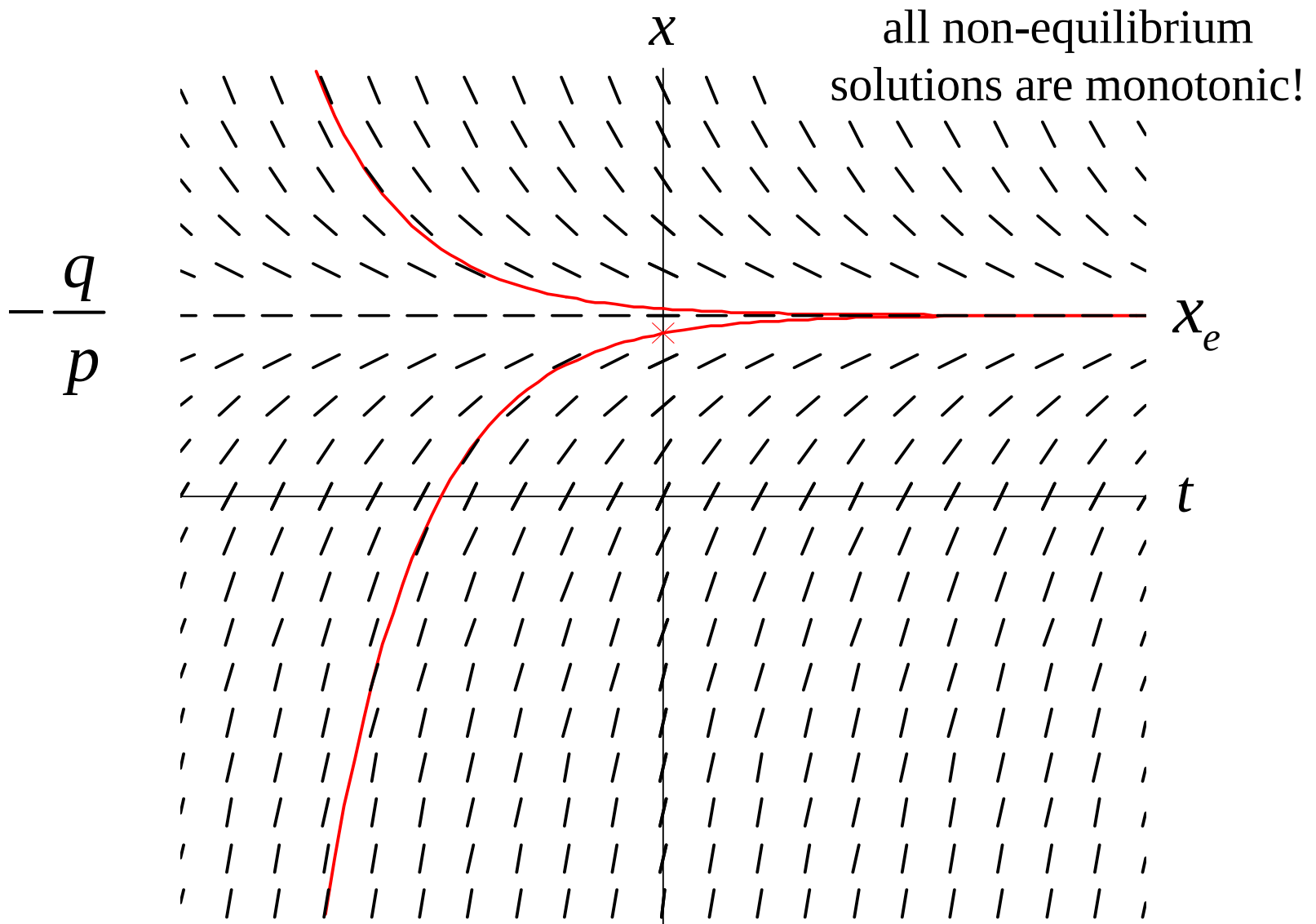
$$x(t) = ce^{pt} - q / p .$$

NOTE:

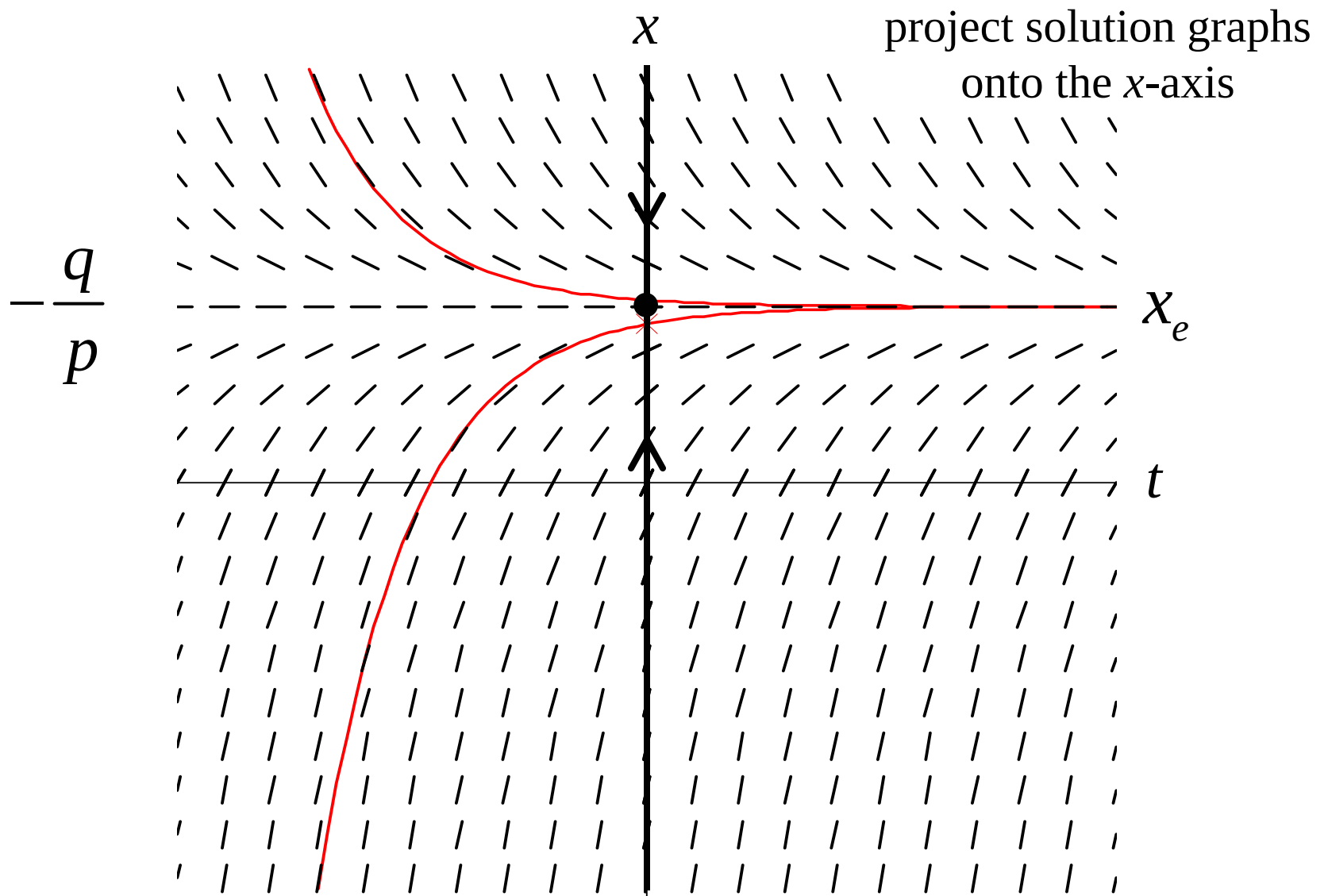
All non-equilibrium solutions are *monotonic*
(i.e., either strictly *increasing* or *decreasing*)
and

$$x(t) = \begin{cases} \text{unbounded if } p > 0 & (\text{unless } x(0) = -q / p) \\ \text{tends to the equilibrium as } t \rightarrow +\infty & \text{if } p < 0 \end{cases}$$

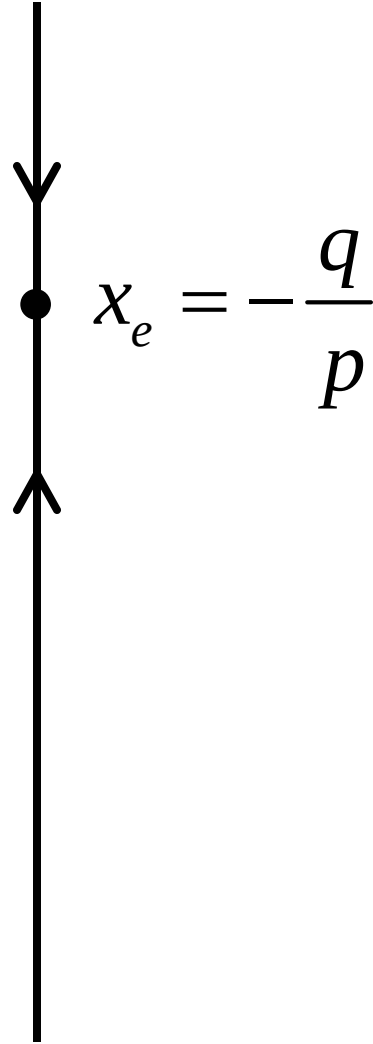
$$x' = px + q, \quad p < 0$$



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$$x' = px + q, \quad p < 0$$



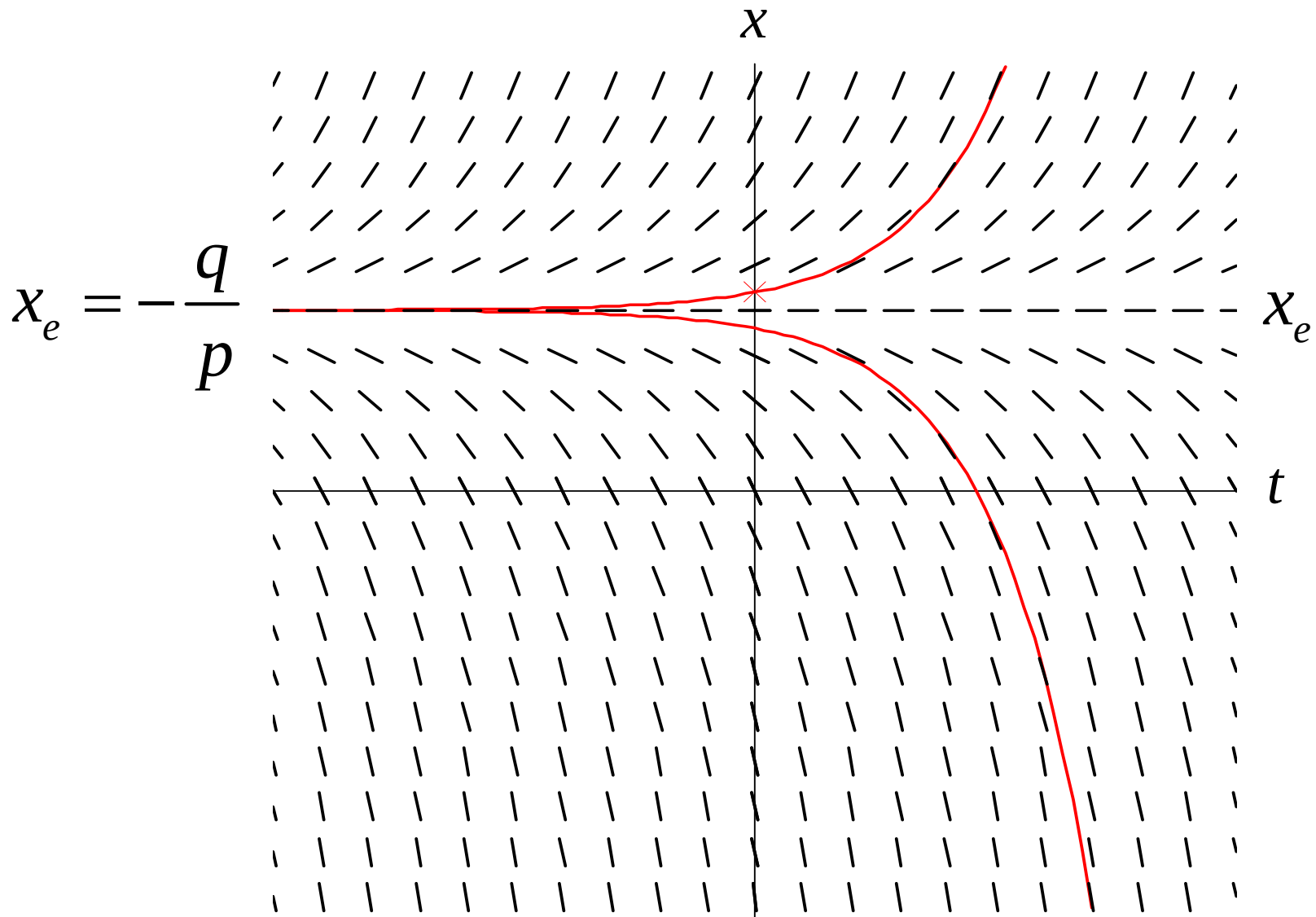
$$x' = px + q, \quad p < 0$$

$$p < 0: \quad \longrightarrow \quad x_e \quad \longleftarrow$$

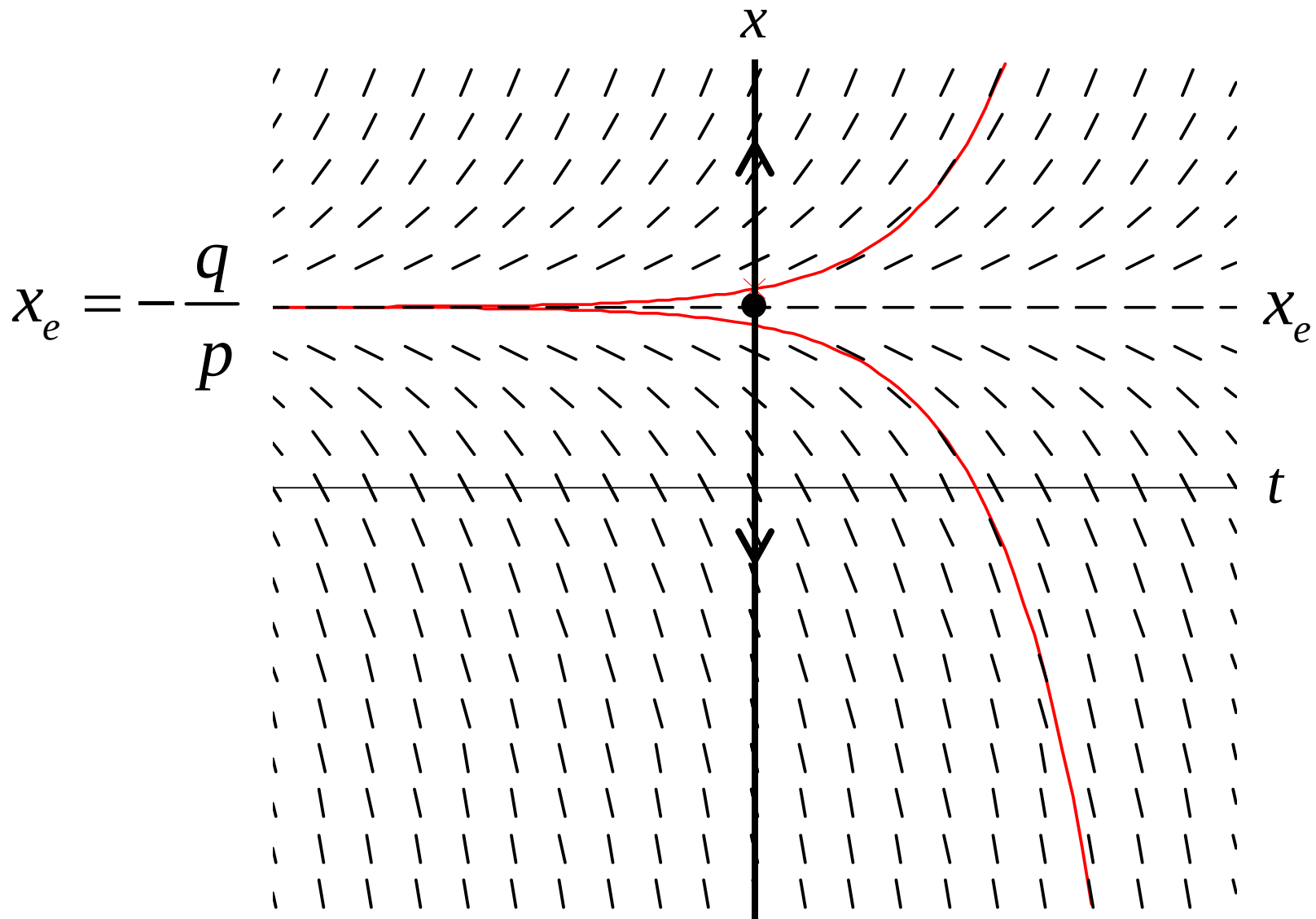
Phase Line Portrait

Called a *sink* (or *stable equilibrium* or an *attractor*)

$$x' = px + q, \quad p > 0$$



$$x' = px + q, \quad p > 0$$



$$x' = px + q, \quad p > 0$$

$$p > 0: \quad \longleftarrow x_e \longrightarrow$$

Called a *source* (or *unstable equilibrium* or a *repeller*)

Phase line portraits
for linear autonomous equations

$p < 0$: $\longrightarrow x_e \longleftarrow$

$p > 0$: $\longleftarrow x_e \longrightarrow$

$p = 0$: called a *bifurcation point*