

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 05-Feb-08

<http://math.arizona.edu/~cushing/250b.html>

1.2.2 Numerical Approximations

Summary from last class

$$x' = x, \quad x(0) = 1$$

To estimate $x(1)$ we used the

Left Hand Rectangle Rule : $\int_a^b f(t, x(t))dt \approx (b-a) f(a, x(a))$

By hand we calculated

using one step: $x(1) \approx 2$

using two steps: $x(1) \approx 2.25$

The computational pattern was

$$x_{i+1} = x_i + s x_i$$

$s =$ step size

$$x' = x, \quad x(0) = 1$$

Euler Algorithm

	Step size s	Approximation to $x(1)$
5 steps	0.200000	2.488320
10 steps	0.100000	2.593742
etc.	0.050000	2.653298
	0.025000	2.685064
	0.012500	2.701485
	0.006250	2.709836
	0.003125	2.714047



note apparent
convergence

1.2.2 Numerical Approximations

$$x' = f(t, x)$$

$$x(t_0) = x_0$$

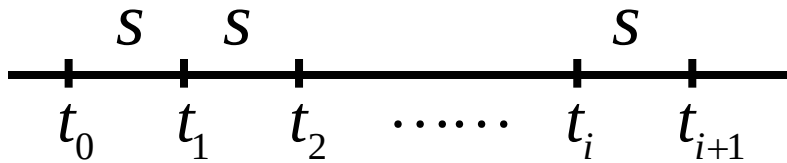
Euler Algorithm

x_0 = the initial condition

$$x_{i+1} = x_i + s f(t_i, x_i)$$

s = step size

x_i approximates $x(t_i)$



Based on the Left Hand Rectangle Rule for integrals

$$\int_a^b f(t, x(t)) dt \approx (b - a) f(a, x(a))$$

Some Facts about the Euler Algorithm

- Under the criteria for the Fundamental Theorem
the Euler approximations *converge* to the solution
as the step-size s tends to 0.
- The errors tend to 0 *at the same rate as s tends to 0*.

Roughly: errors go down by $1/2$ if s goes down by $1/2$

errors go down by $1/10$ if s goes down by $1/10$

etc.

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errors go down by 1/10 if s goes down by 1/10
etc.

Euler's method is said to be “*of order 1*” or $O(s)$.

In practice this can be *too slow* !

Higher order (faster convergent) methods are based on
other integral approximation methods

Heun Algorithm

(Improved or Modified Euler Algorithm)

Order 2 (trapezoid rule)

Runge-Kutta Methods

One for any order you want !

Runge-Kutta 4th order (RK-4) is popular

$$x' = x^2, \quad x(0) = 0.99$$

	Huen	RK-4
Step size s	Approximation to $x(1)$	Approximation to $x(1)$
0.100000	19.346653	53.355933
0.050000	33.073325	75.881773
0.025000	52.217973	91.639594
0.012500	72.662362	97.671604
0.006250	87.787581	98.856123
0.003125	95.273334	98.988718
0.001563	97.719807	98.999238
0.000781	98.719804	98.999951
0.000391	98.928245	98.999997

(Exact answer turns out to be $x(1) = 99$)

$$x' = f(t, x), \quad x(t_0) = x_0$$

We can plot approximated points

$$(t_i, x(t_i)) \approx (t_i, x_i)$$

(using any method) and connect them with straight line segments to obtain *an approximate graph* of the solution.

Graphs will converge as the step-size s decreases to 0.

Use a computer program look at solution graph for

$$x' = x, \quad x(0) = 1$$

and

$$x' = x^2, \quad x(0) = 0.99$$

2.1 & 2.2 Solution of Linear Equations

$$x' = f(t, x)$$

DEFINITION: The equation is *linear* if $f(t, x)$ is linear in x .

$$x' = p(t)x + q(t)$$

Note: This does *not* say the solution $x(t)$ is linear in t !

$$x' = x$$

$$x' = x \sin t$$

$$x' = x^2$$

$$x' = t \sin x$$

$$x' = 2x + 1$$

$$v' = 9.8 - k_0 v$$

$$x' = -3x + \sin t$$

$$v' = 9.8 - k_0 v^2$$

2.1 & 2.2 Solution of Linear Equations

How to find solution formulas

$$x' = p(t)x + q(t)$$

When $q(t) \equiv 0$ the equation is *homogeneous* :

$$x' = p(t)x$$

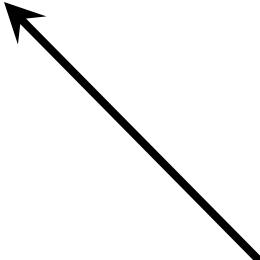
The *general solution* is

$$x(t) = ce^{P(t)}, \quad P(t) = \int p(t)dt, \quad c = \text{arbitrary constant}$$

That is to say,
these are *all* of the solutions



Any anti-derivative
 $P(t)$ will do.



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$$x' = x$$

$$x' = px, \quad p \text{ a constant}$$

$$x' = x \sin t$$

$$x' = t^{-1}x, \quad t > 0$$

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The *general solution* is

$$x(t) = ce^{P(t)}, \quad P(t) = \int p(t)dt, \quad c = \text{arbitrary constant}$$

$$x' = x \quad \Rightarrow x = ce^t$$

$$x' = px \quad \Rightarrow x = ce^{pt}$$

$$x' = x \sin t \Rightarrow x = ce^{-\cos t}$$

$$x' = t^{-1}x \quad \Rightarrow x = ct \quad \text{for } t > 0$$