

MATH 250b – Section 2

Spring Semester 2008

Tuesday, 29-Jan-08

<http://math.arizona.edu/~cushing/250b.html>

Some differential equations we've encountered so far ...

$$x'' = 9.8$$

$$mx'' = mg - cx'$$

$$v' = 9.8$$

$$mv' = 9.8 - cv$$

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Some other examples

$$x' = t$$

$$x' = x$$

$$x' = x^2 + t^2$$

$$x' = rx \left(1 - \frac{x}{K} \right)$$

$$x'' + \alpha(x^2 - 1)x' + x = 0$$

$$\begin{cases} x' = -2x + 2y \\ y' = 2x - 5y \end{cases}$$

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This is an example of a
system of differential equations

2 equations for 2 unknowns

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The *order of an equation* is that of the highest order derivative appearing in the equation

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Second order equations

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First order system

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$$x'' = 9.8$$

$$mx'' = mg - cx'$$

$$v' = 9.8$$

$$mv' = 9.8 - cv$$

Some other examples

$$x' = t$$

It turns out that

$$x' = x$$

Second (and higher) order equations

$$x' = x^2 + t^2$$

can be converted to first order systems

(and vice versa)

$$x' = rx \left(1 - \frac{x}{K} \right)$$

$$x'' + \alpha(x^2 - 1)x' + x = 0$$

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Organizational Structure of the Course

First Order Equations (Chapters 1 - 3)

- | | | |
|------------------------|---|-------------------------|
| 1. Linear Equations | { | other fundamental |
| 2. Nonlinear Equations | | classification criteria |

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First Order Systems (Chapters 4 - 8)

- | | | |
|----------------------|---|-----------------|
| 1. Linear Systems | { | includes higher |
| 2. Nonlinear Systems | | order equations |

The list of topics studied for first order equations
will be re-visited (in same order) for first order systems

This week we will cover

Chapter 1: First Order Equations
(basic theory)

- When can be sure a differential equation has solutions ?

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(basic theory)

- When can be sure a differential equation has solutions ?

Since formulas for solutions can rarely be found,
we want to study

- some basic graphic & numerical approximations to solutions.

SOLUTIONS

Examples

$$x' = t$$

has solutions $x(t) = \frac{1}{2}t^2 + c$

where c is an arbitrary constant

$$x' = x$$

has solutions $x(t) = ce^t$
where c is an arbitrary constant

Substitution reduces differential equation to an identity in t

$$\text{LHS} = x'(t) = \frac{d}{dt} \left(\frac{1}{2}t^2 + c \right) = t$$

$$\text{RHS} = t$$

Identical for all t

$$\text{LHS} = x'(t) = \frac{d}{dt} (ce^t) = ce^t$$

$$\text{RHS} = x(t) = ce^t$$

Identical for all t

SOLUTIONS

Example

$$x' = 2tx^2$$

$$\text{has a solution } x(t) = \frac{1}{1-t^2}$$

Substitution reduces differential equation to an identity in t

$$\text{LHS} = x'(t) = \frac{d}{dt} \left(\frac{1}{1-t^2} \right) = -\frac{-2t}{(1-t^2)^2} = \frac{2t}{(1-t^2)^2}$$

$$\text{RHS} = 2t(x(t))^2 = 2t \left(\frac{1}{1-t^2} \right)^2 = \frac{2t}{(1-t^2)^2}$$

Identical for all t *on the interval* $-1 < t < 1$

or on the interval

$$t < -1 \quad \text{or} \quad t > 1$$

SOLUTIONS

$$x' = f(t, x)$$

(normal form of a first order equation)

By a *solution on an interval* $a < t < b$ we mean
a function $x = x(t)$ defined and differentiable on $a < t < b$
that reduces the differential equation
to an identity on the interval $a < t < b$.

That is to say,

$$x'(t) \equiv f(t, x(t))$$

for all t from the interval $a < t < b$.

The Fundamental Existence & Uniqueness Theorem

$$x' = f(t, x)$$

Consider the right-hand-side $f(t, x)$ as a function of its arguments t and x .

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$$x' = f(t, x)$$

Consider the right-hand-side $f(t, x)$ as a function of its arguments t and x .

$$x' = t \Rightarrow f(t, x) = t$$

$$x' = x \Rightarrow f(t, x) = x$$

$$x' = x^2 + t^2 \Rightarrow f(t, x) = x^2 + t^2$$

$$x' = r x \left(1 - \frac{x}{K}\right) \Rightarrow f(t, x) = r x \left(1 - \frac{x}{K}\right)$$

$$x' = 2tx^2 \Rightarrow f(t, x) = \frac{1}{1-t^2}$$

The Fundamental Existence & Uniqueness Theorem

The initial value problem IVP:

$$x' = f(t, x)$$

$$x(t_0) = x_0$$

If

(1) $f(t, x)$ is continuous and

(2) its derivative $\frac{df(t, x)}{dx}$ is continuous

(at $t = t_0$ and $x = x_0$)

then the initial value problem has a solution

(on an interval containing t_0) and this solution is unique.

EXAMPLE

$$x' = \sqrt{1 - x^2}$$

$$x(0) = 0$$

Text, page 25

Exercise 27

$$f(t, x) = \sqrt{1 - x^2}$$

continuous for $t \approx 0$ and $x \approx 0$

$$\frac{\partial f(t, x)}{\partial x} = \frac{-2x}{\sqrt{1 - x^2}}$$

continuous for $t \approx 0$ and $x \approx 0$

Fundamental Theorem implies there exists a solution
on an interval containing $t = 0$
and this solution is unique.