

MATH 250b – Section 2

Spring Semester 2008

Thursday, 24-Jan-08

<http://math.arizona.edu/~cushing/250b.html>

A Model with Friction

3. *An object, dropped from a high location, falls vertically.*

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

Step 1: Model Derivation

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

$x = x(t)$ = distance fallen, t = time

m = object's mass

c = coefficient of friction

g = acceleration due to gravity

A Model with Friction

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Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc.

Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

Step 1: Model Derivation

Governing laws
(assumptions / hypotheses) $\rightarrow mx''(t) = mg - cx'(t)$

$x(0) = 0$ $\leftarrow$ Reference point

$x'(0) = 0$ $\leftarrow$ Object is dropped

$x = x(t)$ = distance fallen, t = time

m = object's mass

c = coefficient of friction > 0

g = acceleration due to gravity

A Model with Friction

3. An object, dropped from a high location, falls vertically.

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc.

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(assumptions / hypotheses) $\rightarrow mx''(t) = mg - cx'(t)$

$x(0) = 0$ $\leftarrow$ Reference point

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Model variables & parameters (coefficients)

Unknown
("state" variable) $\rightarrow x = x(t) = \text{distance fallen}, \quad t = \text{time}$

$m = \text{object's mass}$

$c = \text{coefficient of friction}$

$g = \text{acceleration due to gravity}$

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

This is an example of an
“*Ordinary Differential Equation*”

with “*initial conditions*”

... an “*initial value problem*”

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

Remark

Sometimes the equation is re-written as

$$x''(t) = g - \frac{c}{m} x'(t)$$

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

Remark

Sometimes the equation is re-written as

$$x''(t) = g - \boxed{\frac{c}{m}}x'(t)$$

per unit mass
coefficient of friction

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

We are not yet in a position to perform the
solution step.

We will return to this problem

... to see whether or not it does a better job
predicting the experimental data.

Solution
Step

$$x(t) = ??$$

**MATHEMATICAL
SOLUTION**

Re-do this problem using *velocity* as unknown (state) variable:

4. *An object, dropped from a high location, falls vertically.*

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mv'(t) = mg - cv(t)$$

$$v(0) = 0$$

or

$$v'(t) = g - \frac{c}{m}v(t)$$

$$v(0) = 0$$

Solution
Step

??

$$v(t) = ??$$

**MATHEMATICAL
SOLUTION**

The math problems from these applications involve ordinary differential equations for an unknown function with supplementary initial conditions :

$$x''(t) = 9.8$$

$$x(0) = 0$$

$$x'(0) = 0$$

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

$$v'(t) = 9.8$$

$$v(0) = 0$$

$$mv'(t) = 9.8 - cv(t)$$

$$v(0) = 0$$

The math problems from these applications involve ordinary differential equations for an unknown function with supplementary initial conditions :

$$x''(t) = 9.8$$

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$$x(0) = 0$$

$$x'(0) = 0$$

$$v'(t) = 9.8$$

$$v(0) = 0$$

$$mv'(t) = 9.8 - cv(t)$$

$$v(0) = 0$$

- Do *initial value problems* like these always have solutions ?
- What exactly is meant by “solution” ?
- How many solutions can they have ?
If more than one, what does it mean for our applications ?

Step 1 practice problems (from text)

- (p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*
- (a) *Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

Step 1 practice problems (from text)

(p. 16, #1) Suppose the balance in a savings account grows, at each instant of time, at a **rate** proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.

(a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

Key words: tells you a derivative is involved

Step 1 practice problems (from text)

- (p. 16, #1) Suppose the balance in a savings account grows, at each instant of time, at a rate **proportional to** the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.
- (a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .
- 

Key words: Means “**is a constant multiple of**”

Requires a symbol for the “constant of proportionality”

Step 1 practice problems (from text)

(p. 16, #1) Suppose the balance in a savings account grows, at each instant of time, at a **rate proportional to the balance** present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.

(a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .


$$x'(t) = r x(t)$$

Step 1 practice problems (from text)

(p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a **rate proportional to the balance** present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*

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$$x'(t) = r x(t)$$

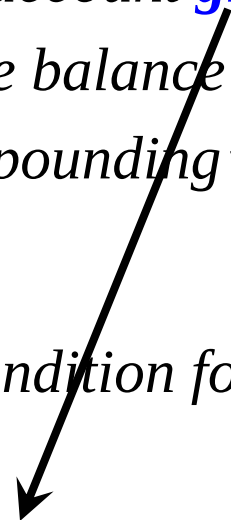
r = constant of proportionality

What is its sign ?

What are its units ?

Step 1 practice problems (from text)

- (p. 16, #1) Suppose the balance in a savings account **grows**, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.
- (a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

$$x'(t) = r x(t), \quad r > 0$$


Step 1 practice problems (from text)

(p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*

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$$x'(t) = r x(t), \quad r > 0$$

What are the units of r ?

Step 1 practice problems (from text)

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$$x'(t) = r x(t), \quad r > 0$$

What are the units of r ?

Units on both sides must be the same (i.e., must “balance”).

Step 1 practice problems (from text)

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$\frac{\text{dollars}}{\text{time}}$

dollars

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$$x'(t) = r x(t), \quad r > 0$$

What are the units of r ?

Units on both sides must be the same (i.e., must “balance”).

$$\frac{\text{dollars}}{\text{time}} = \frac{1}{\text{time}} \times \text{dollars}$$

Step 1 practice problems (from text)

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(a) *Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

$$x'(t) = r x(t), \quad r > 0$$

What are the units of r ?

Units on both sides must be the same (i.e., must “balance”).

$$r = (\text{per unit or per capita}) \text{ growth rate} = \frac{x'(t)}{x(t)}$$

Step 1 practice problems (from text)

(p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*

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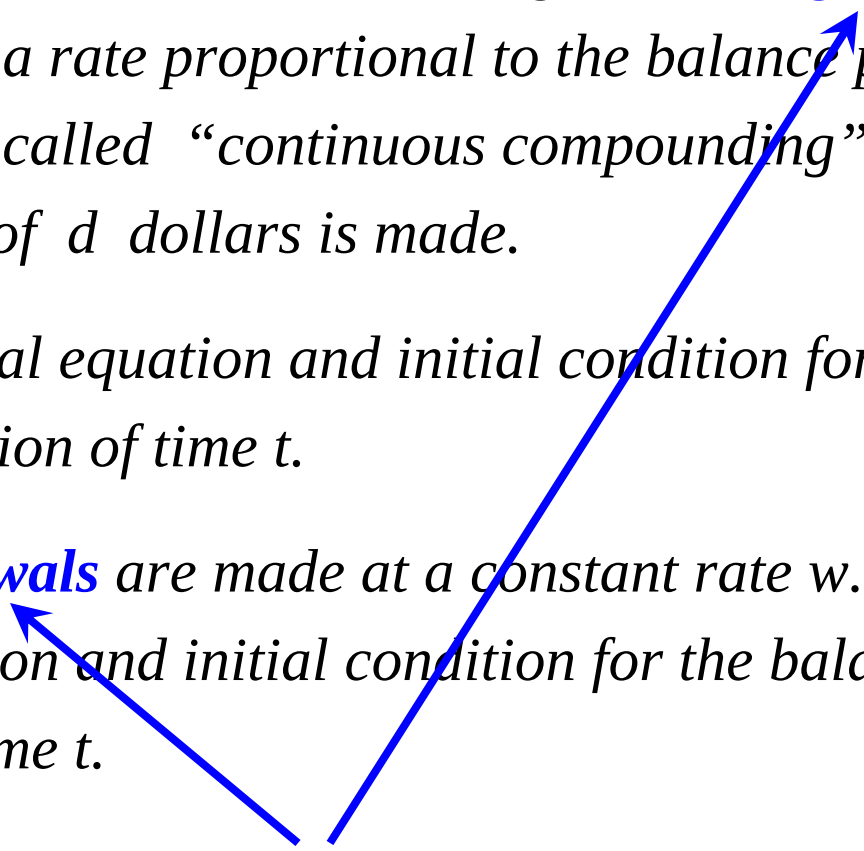
$$x'(t) = r x(t), \quad r > 0$$

$$x(0) = d$$

Step 1 practice problems (from text)

- (p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*
- (a) *Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*
- (b) *Suppose withdrawals are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

Step 1 practice problems (from text)

- (p. 16, #1) Suppose the balance in a savings account **grows**, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.
- (a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .
- (b) Suppose **withdrawals** are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .
- 

Key words:

General Rate Balance Law

total rate of change = input rate – out put rate

Step 1 practice problems (from text)

- (p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*
- (a) *Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*
- (b) *Suppose withdrawals are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

Balance rate of change = growth rate – withdrawal rate

Step 1 practice problems (from text)

(p. 16, #1) Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.

(a) Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

(b) Suppose withdrawals are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .

$$x'(t) = r x(t) - w, \quad r, w > 0$$

$$x(0) = d$$

Note

w has units of
 $\frac{\text{dollars}}{\text{time}}$

Step 1 practice problems (from text)

(p. 16, #1) *Suppose the balance in a savings account grows, at each instant of time, at a rate proportional to the balance present at that time. (This is called “continuous compounding”.) Suppose an initial deposit of d dollars is made.*

(a) *Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

(b) *Suppose withdrawals are made at a constant rate w . Write a differential equation and initial condition for the balance $x = x(t)$ as a function of time t .*

$$\begin{aligned}x'(t) &= r x(t) - w, \quad r, w > 0 \\x(0) &= d\end{aligned}$$

Note

w is not a
per unit rate

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in **temperature** between the body and its **surrounding environment**. Write a differential equation for the **temperature of the body**.*

Key words: basic parameters needing symbolization

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the **rate** at which a body cools **is proportional to** the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

Key words: “derivative of body temperature is a multiple of ”

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

$x(t)$ = temperature of body

t = time

b = temperature of environment

$$x'(t) = c(x(t) - b)$$

What is the sign of c ? Units ?

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

$x(t)$ = temperature of body

t = time

b = temperature of environment

$$x'(t) = c(x(t) - b)$$

$$x(t) - b > 0 \Rightarrow x'(t) < 0$$

body warmer than environment implies it should cool

$$\Rightarrow c < 0$$

Step 1 practice problems (from text)

(p. 16, #9) *Newton's Law of Cooling states that the rate at which a body cools is proportional to the difference in temperature between the body and its surrounding environment. Write a differential equation for the temperature of the body.*

$x(t)$ = temperature of body

t = time

b = temperature of environment

$$x'(t) = -a(x(t) - b), \quad a > 0$$

Step 1 practice problems (from text)

(p. 16, #5) *Suppose a population has a constant per unit death rate $d > 0$ and a constant per unit birth rate $b > 0$.*

(a) Write a differential equation for the population size $x(t)$.

Step 1 practice problems (from text)

(p. 16, #5) Suppose a population has a constant per unit **death rate** $d > 0$ and a constant per unit **birth rate** $b > 0$.

(a) Write a differential equation for the population size $x(t)$.

Key words:

Rate Balance Law

population rate of change = birth rate – death rate

Step 1 practice problems (from text)

(p. 16, #5) *Suppose a population has a constant per unit death rate $d > 0$ and a constant per unit birth rate $b > 0$.*

(a) Write a differential equation for the population size $x(t)$.

$$\text{birth rate} = bx(t)$$

$$\text{death rate} = d x(t)$$

$$x'(t) = bx(t) - d x(t)$$

Step 1 practice problems (from text)

(p. 16, #5) Suppose a population has a constant per unit death rate $d > 0$ and a constant per unit birth rate $b > 0$.

(a) Write a differential equation for the population size $x(t)$.

(b) Suppose the per unit death rate d is not constant, but is instead *proportional to population* size. Write a differential equation for the population size $x(t)$.

Step 1 practice problems (from text)

(p. 16, #5) Suppose a population has a constant per unit death rate $d > 0$ and a constant per unit birth rate $b > 0$.

(a) Write a differential equation for the population size $x(t)$.

(b) Suppose the per unit death rate d is not constant, but is instead *proportional to population* size. Write a differential equation for the population size $x(t)$.

$$d = cx(t), \quad c > 0$$

$$\text{birth rate} = bx(t)$$

$$\text{death rate} = cx^2(t)$$

$$x'(t) = bx(t) - cx^2(t)$$