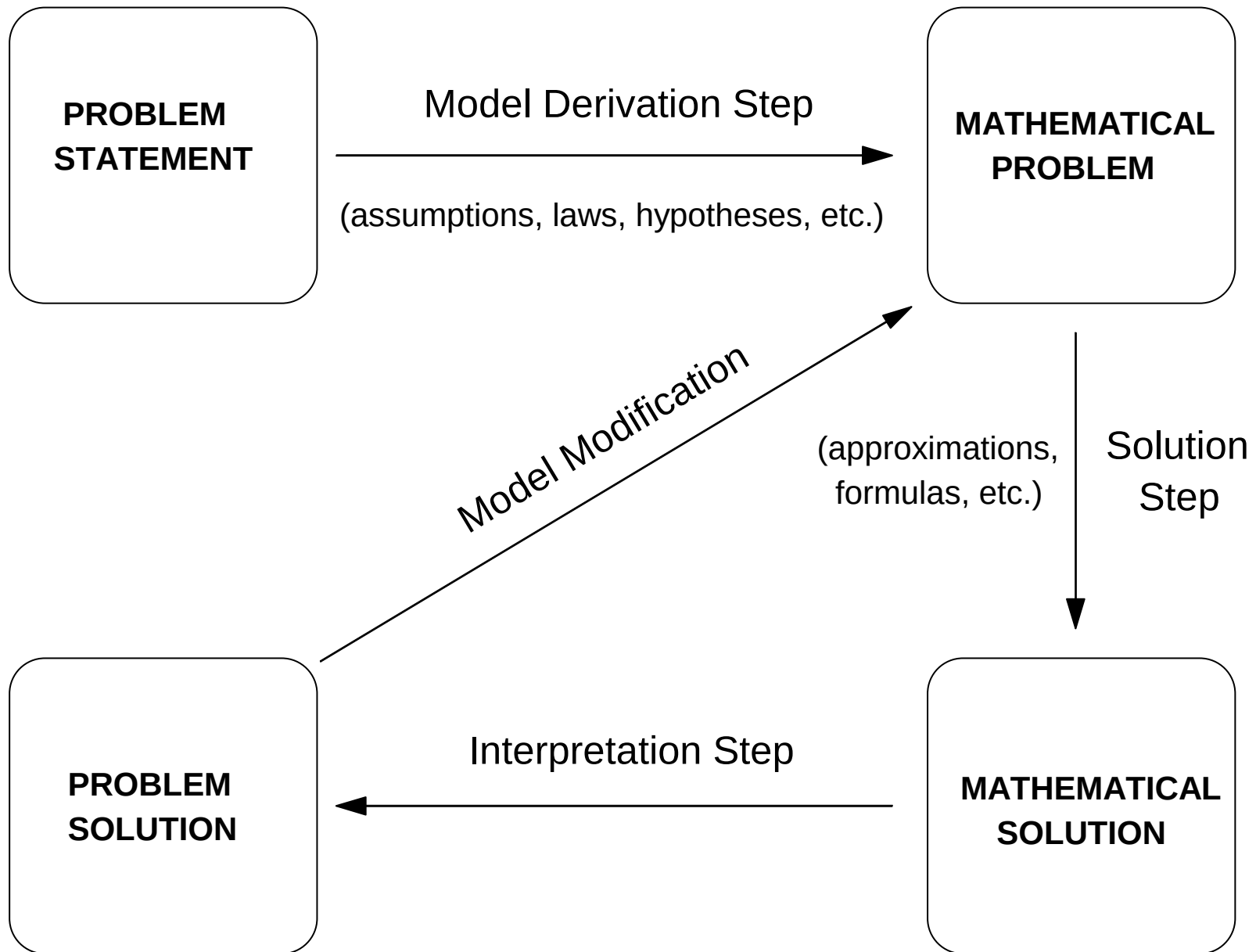


MATH 250b – Section 2

Spring Semester 2008

Tuesday, 22-Jan-08

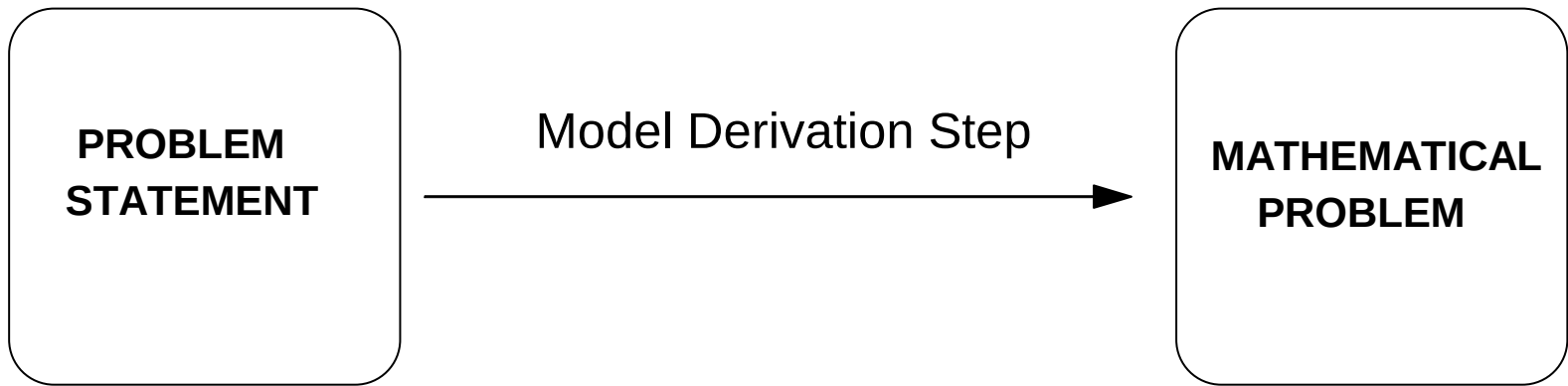
<http://math.arizona.edu/~cushing/250b.html>



Review ...

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?



Identification of quantities, variables, etc. & their symbolization

x = distance (meters) from drop point, t = time (seconds)

a = acceleration (m/s^2)

Classify and relate quantities & variables

$x = x(t)$ = dependent variable, t = independent variable

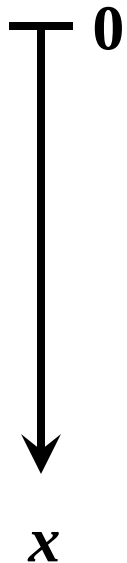
$a = x''(t)$, a relevant "parameter" = 9.8 (*meters / sec²*)

Derive equation for "state variable"

$x''(t) = 9.8$ (directly from problem statement)

$x(0) = 0$ (our choice of reference point)

$x'(0) = 0$ (mathematical description of "dropped")



**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

$$x''(t) = 9.8$$

$$x(0) = 0$$

$$x'(0) = 0$$

Step 1: results in a mathematical problem
for the unknown (function) x .

**PROBLEM
STATEMENT**

Model Derivation Step

$$x''(t) = 9.8$$

$$x(0) = 0$$

$$x'(0) = 0$$

**MATHEMATICAL
PROBLEM**

Step 1: results in a mathematical problem
for the unknown (function) x .

Step 2: solve model equations for x .

Solution
Step

$$x(t) = 4.9t^2$$

**MATHEMATICAL
SOLUTION**

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

$$x''(t) = 9.8$$

$$x(0) = 0$$

$$x'(0) = 0$$

Step 1: results in a mathematical problem
for the unknown (function) x .

Step 2: solve model equations for x .

Step 3: use solution x an original questions.

Solution
Step

$$x(1) = 4.9m$$

$$x(5) = 122.5m$$

$$x(30) = 4,410m$$

**PROBLEM
SOLUTION**

Interpretation Step

$x(t) = 4.9t^2$
**MATHEMATICAL
SOLUTION**

In this problem, perhaps better
termed a "Utilization Step"

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

A challenge:

Solve this problem using velocity v
instead of position x
as the unknown (state) variable

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

Step 1

Identify variables & symbolize

$v =$ velocity (m/sec)

$t =$ time (sec)

$x =$ distance fallen (m)

$t = 0$ = release time

$x(0) = 0$ = reference point

Relate variables

$v = v(t)$

$x = x(t)$

$x'(t) = v(t)$

Translate

$v'(t) = 9.8$

$v(0) = 0$

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

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Translate

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Our choice of state variable

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

Step 1

Identify variables & symbolize

$v =$ velocity (m/sec)

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$x = x(t)$

$x'(t) = v(t)$

Translate

$v'(t) = 9.8$

$v(0) = 0$

Suggested by question

1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

Step 1

Identify variables & symbolize

$v =$ velocity (m/sec)

$t =$ time (sec)

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$x(0) = 0 =$ reference point

Relate variables

$v = v(t)$

$x = x(t)$

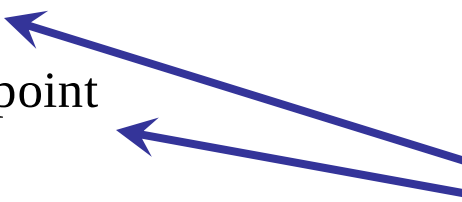
$x'(t) = v(t)$

Translate

$v'(t) = 9.8$

$v(0) = 0$

Our choice of
reference points



1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

Step 1

Identify variables & symbolize

$v =$ velocity (m/sec)

$t =$ time (sec)

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Relate variables

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$x'(t) = v(t)$

Translate

$v'(t) = 9.8$

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1. An object, dropped from a high location, falls vertically with a constant acceleration of 9.8 meters/sec^2 .

How far will it fall in 1 second ? 5 seconds ? 1/2 minute ?

Step 1: Model Derivation

Model equations for unknown $v(t)$

$$v'(t) = 9.8$$

$$v(0) = 0$$

Step 2: Solution of Model Equations

Solve for $v(t) = 9.8t$

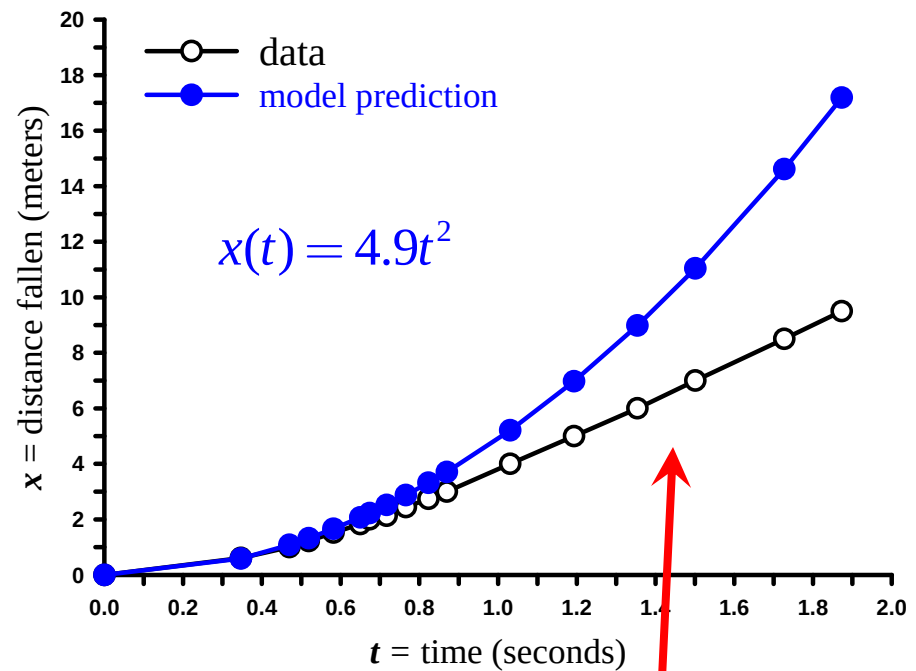
Step 3: Use Solution to Answer Question

Calculate $x(t) = 4.9t^2$

Same answers as before

Data from a lab experiment*

t (seconds)	x (meters)	x predicted (meters)	% error
0.000	0.00	0.00	--
0.347	0.61	0.59	-3.28
0.470	1.00	1.08	8.24
0.519	1.22	1.32	8.16
0.582	1.52	1.66	9.19
0.650	1.83	2.07	13.13
0.674	2.00	2.23	11.30
0.717	2.13	2.52	18.26
0.766	2.44	2.88	17.83
0.823	2.74	3.32	21.13
0.870	3.00	3.71	23.63
1.031	4.00	5.21	30.21
1.193	5.00	6.98	39.48
1.354	6.00	8.98	49.72
1.501	7.00	11.04	57.71
1.727	8.50	14.61	71.93
1.873	9.50	17.19	80.95



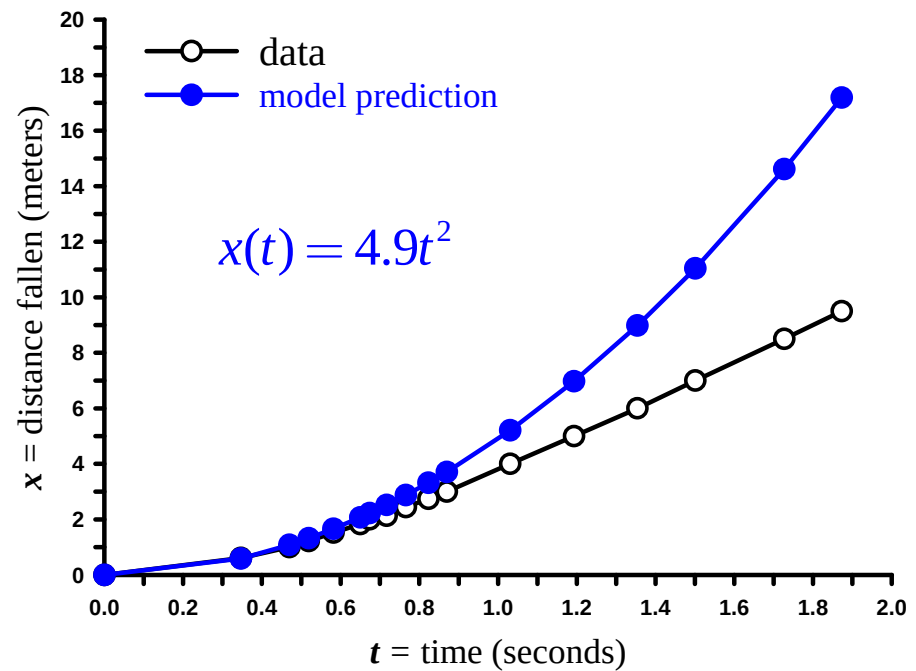
Problem

How to account for **discrepancy** between model predictions & observed data ?

* M. Peastrel, R. Lynch and A. Armenti, Jr., 1993, *The Physics of Sports* (A. Armenti, Jr., editor), American Institute of Physics, New York, pp. 301-303

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Some possibilities

Mathematical errors

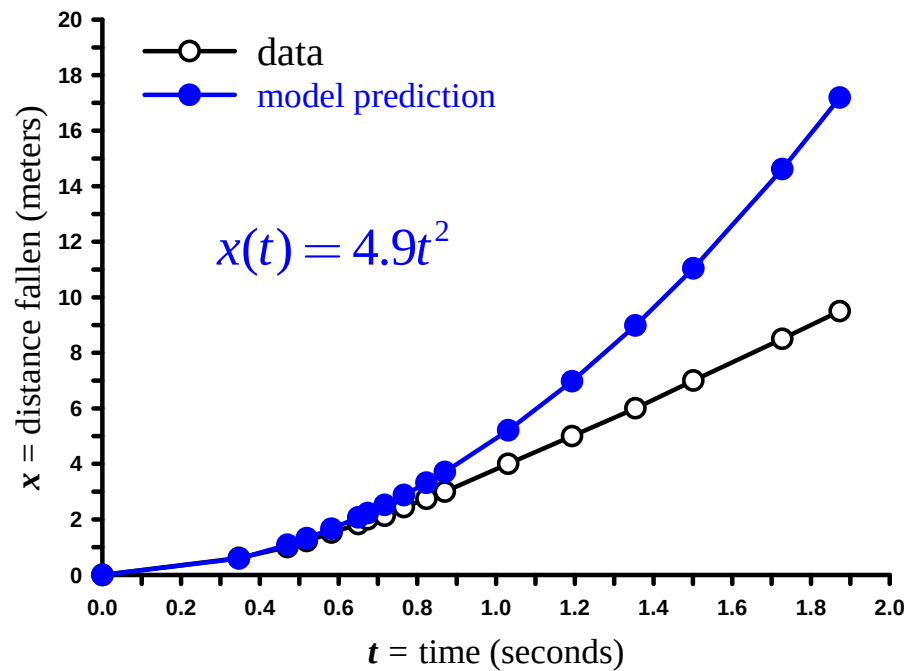
Experimental errors

Model is “wrong” - overlooked/ignored
important mechanism(s)

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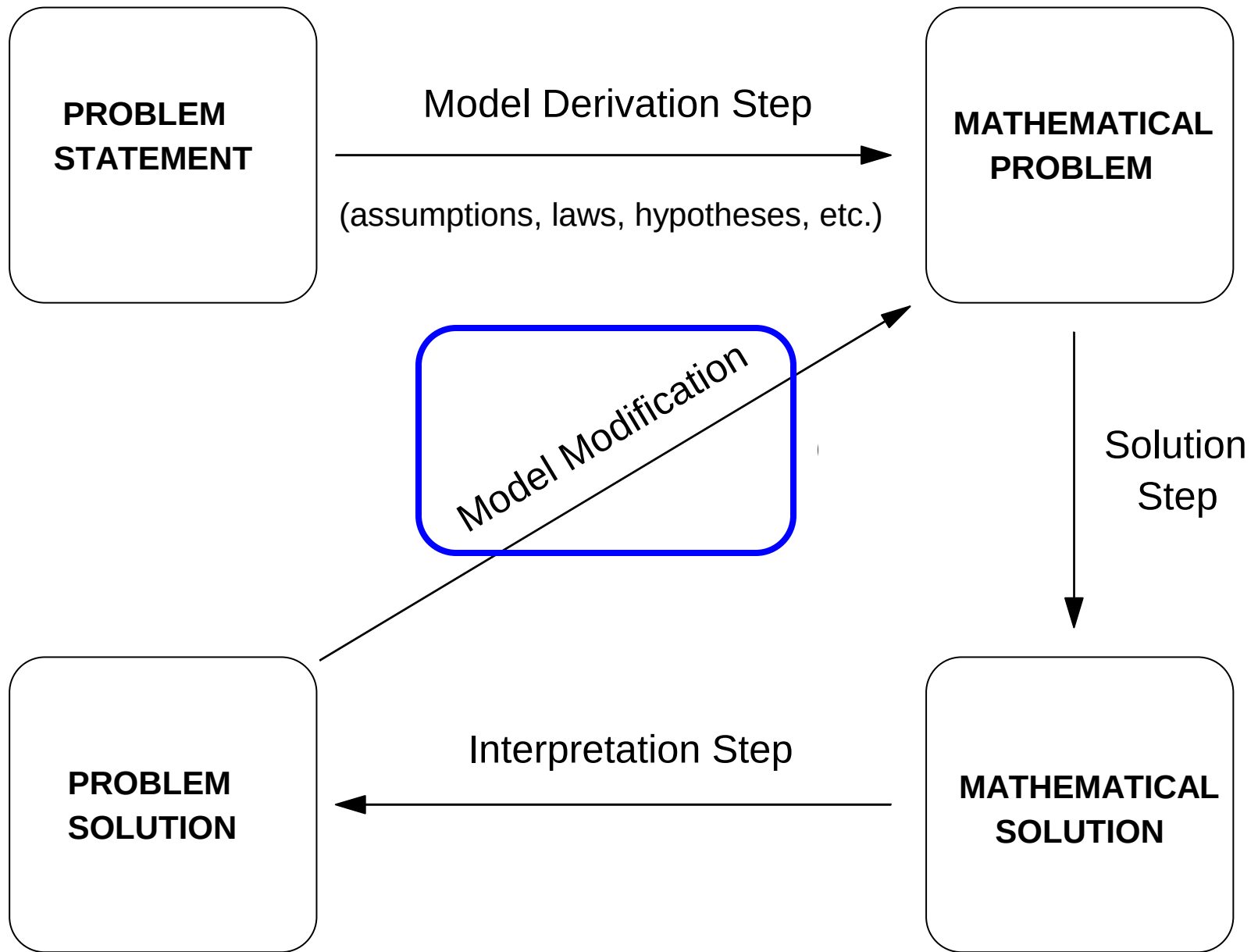


Turns out the experiment was done with a *badminton shuttlecock*.

By design, shuttlecocks are subject to considerable *air resistance/friction*.

We need to *modify our model* so as to take friction into account.

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A Model with Friction

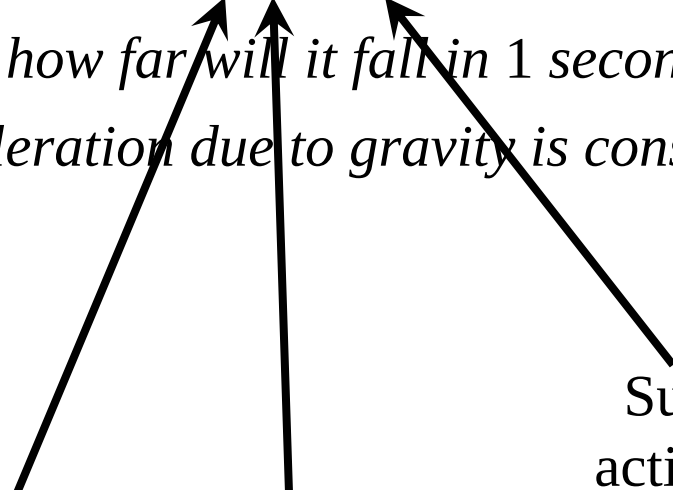
3. *An object, dropped from a high location, falls vertically.*

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

A Model with Friction

3. An object, dropped from a high location, falls vertically.

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second? 5 seconds? etc. Assume acceleration due to gravity is constant $g (= 9.8 \text{ m/s}^2)$.



object's mass & acceleration

The diagram consists of three arrows originating from labels at the bottom and pointing to terms in the equation $ma = F$ in the text above. One arrow points from 'object's mass & acceleration' to 'm'. A second arrow points from the same label to 'a'. A third arrow points from 'Sum of all forces acting on the object' to 'F'.

Sum of all forces
acting on the object

A Model with Friction

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Force of friction
is proportional to velocity



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Force of friction

“ is proportional to ” velocity

means

“ is a constant multiple of ”

A Model with Friction

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Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant $g (= 9.8 \text{ m/s}^2)$.

Our approach is to modify previous model, so we adopt the same notation/variables, etc.

$x(t)$ = distance fallen

t = time

$v(t) = x'(t)$ = velocity

$x(0) = 0, \quad x'(0) = 0$

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$$mx''(t) =$$

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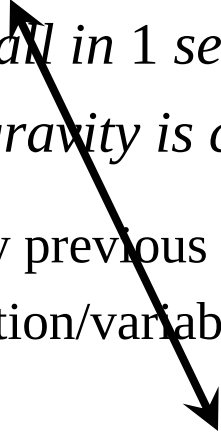
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$$mx''(t) = F_{\text{gravity}} + F_{\text{friction}}$$

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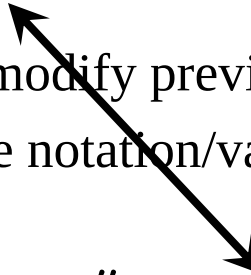
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$$mx''(t) = mg + F_{\text{friction}}$$

A Model with Friction

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$$mx''(t) = mg + kx'(t)$$

k = a constant of proportionality

A Model with Friction

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k = a constant of proportionality

Force of friction acts in *opposite direction* of motion :

$F_{\text{friction}} = kx'(t)$ and $x'(t)$ must have opposite signs

that is to say, k must be negative

A Model with Friction

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Our approach is to modify previous model, so we adopt the same notation/variables, etc.

$x(t)$ = distance fallen

t = time

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$x(0) = 0, \quad x'(0) = 0$


$$mx''(t) = mg + kx'(t)$$

$$k < 0$$

Generally, however, we like to have letters represent *positive* parameters

This is so that the signs we see in model equations are really the signs present.

A Model with Friction

3. An object, dropped from a high location, falls vertically.

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant $g (= 9.8 \text{ m/s}^2)$.

Our approach is to modify previous model, so we adopt the same notation/variables, etc.

$x(t)$ = distance fallen

t = time

$v(t) = x'(t)$ = velocity

$x(0) = 0, \quad x'(0) = 0$

$$mx''(t) = mg - cx'(t)$$

$$c > 0$$

Generally, however, we like to have letters represent *positive* parameters

This is so that the signs we see in model equations are really the signs present.

For this reason, we rename the constant of proportionality as $-c$ where $c > 0$

A Model with Friction

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Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc. Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

Step 1: Model Derivation

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

$x = x(t)$ = distance fallen, t = time

m = object's mass

c = coefficient of friction

g = acceleration due to gravity

A Model with Friction

3. An object, dropped from a high location, falls vertically.

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc.

Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

Step 1: Model Derivation

Governing laws
(assumptions / hypotheses)


$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0 \quad \longleftarrow \quad \text{Reference point}$$

$$x'(0) = 0 \quad \longleftarrow \quad \text{Object is dropped}$$

$x = x(t)$ = distance fallen, t = time

m = object's mass

c = coefficient of friction

g = acceleration due to gravity

A Model with Friction

3. An object, dropped from a high location, falls vertically.

Use Newton's Law $ma = F$ and the Linear Friction Law to determine how far will it fall in 1 second ? 5 seconds? etc.

Assume acceleration due to gravity is constant g ($= 9.8 \text{ m/s}^2$).

Step 1: Model Derivation

Governing laws
(assumptions / hypotheses) $\rightarrow mx''(t) = mg - cx'(t)$

$x(0) = 0$ $\leftarrow$ Reference point

$x'(0) = 0$ $\leftarrow$ Object is dropped

Model variables & parameters (coefficients)

Unknown
("state" variable) $\rightarrow x = x(t) = \text{distance fallen}, \quad t = \text{time}$

$m = \text{object's mass}$

$c = \text{coefficient of friction}$

$g = \text{acceleration due to gravity}$

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

This is an example of an
“*Ordinary Differential Equation*”

with “*initial conditions*”

... an “*initial value problem*”

**PROBLEM
STATEMENT**

Model Derivation Step

**MATHEMATICAL
PROBLEM**

new model

$$mx''(t) = mg - cx'(t)$$

$$x(0) = 0$$

$$x'(0) = 0$$

We are not yet in a position to perform the
solution step.

We will return to this problem

... to see whether or not it does a better job
predicting the experimental data.

Solution
Step

$$x(t) = ??$$

**MATHEMATICAL
SOLUTION**