

PROJECT #1 - Math 250b (Spring 2008) - Professor J. M. Cushing
Due date: March 27

Part 1 of this project includes readings from case studies in the text book together with some relevant exercises. Part 2 is a follow-on to the population dynamics case study in Part 1 that involves your modifying or extending the text's modeling effort to include more recent data.

Please do not simply provide answers to the exercises the way you do for the weekly homework. We ask that you give well written, concise but clear, descriptions and explanations of what you have done. Treat your write-up as a report to, say, a supervisor or client in which you must not only provide answers to the questions, but descriptions, explanations, and rationales for the aims and goals of the over all project, the assumptions that underlie the model, the analytic techniques used, the conclusions, any shortcomings/problems/difficulties that might be present, and so on. We will grade the project not only on the basis of the answers you provide, but on your communication skills.

PART 1

- (1) Read Chapter 1, Section 3.6.3, page C-21 through C-26: Flow and Reaction Problems

Do exercise #23(a-e), page C-29 (a nonlinear autonomous equation from an enzyme-substrate reaction problem involving Michaelis-Menten uptake rates)

Do exercise #26(a-b), page C-29 (a modification of the model in exercise #23 that involves a different nonlinear equation arising from Hill's formula for uptake rates)

- (2) Read Chapter 1, Section 3.6.1, through page 135 through C-5: Population dynamics

Do exercise #2 on page C-26. This is a variant of the first pass through the modeling cycle (the exponential growth model).

Do exercise #4 on page C-27. This is a variant of the second pass through the modeling cycle (the doomsday model).

PART 2

Table 1 below shows more recent world population data than is used in Section 3.6.1 of the text.

$t = \text{year}$	$x = \text{world population numbers (billions)}$
1900	1.65
1910	1.75
1920	1.86
1930	2.07
1940	2.30
1950	2.52
1960	3.02
1970	3.70
1980	4.45
1990	5.30
1995	5.68
2000	6.07
2005	6.45

TABLE 1

(1) Plot the data in Table 1 together with the predictions of the doomsday model (6.9) on page C-4 of the text (using the scales in Figure 1). The doomsday model in the text utilizes data from 1900 to 1990. How well does that model predict the world population numbers observed in the subsequent years 1995, 2000 and 2005? What does the subsequent data suggest about the doomsday model?

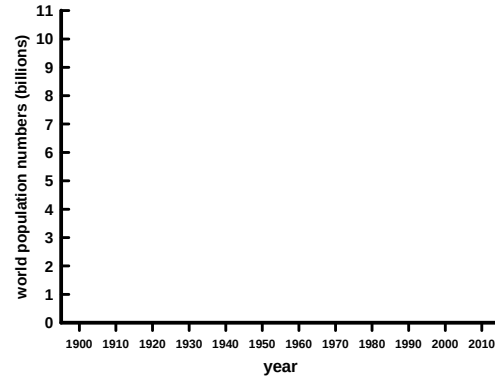


FIGURE 1

(2) Use the data in Table 1 to approximate the per capita rate of change $x'(t)/x(t)$ of the world population by calculating the numbers $\delta(t)$ according to the formula

$$\delta(t) = \frac{1}{x(t)} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

for years $t = 1910$ to 2005 . Place your answers in a table of the below. Plot the data in Table 2, placing x on the horizontal axis and δ on the vertical axis. (Suggestion: use a vertical scale from 0 to 0.025.)

t	x (billions)	δ ($\times 10^{-3}$)
1910	1.75	
1920	1.86	
1930	2.07	
1940	2.30	
1950	2.52	
1960	3.02	
1970	3.70	
1980	4.45	
1990	5.30	
1995	5.68	
2000	6.07	
2005	6.45	

TABLE 3

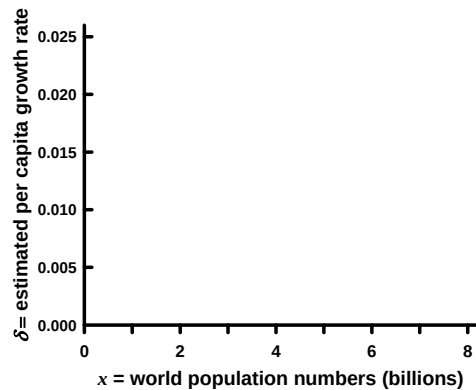


FIGURE 2

(3) (a) Find a differential equation model for $x(t)$ of the form $\frac{x'}{x} = mx + b$ by calculating the linear least squares best fit for $mx + b$ from the data in Table 2. (That is, do a “linear regression” of δ on x to obtain numerical values for m and b .) Plot a graph of your calculated line together with the data in Table 2 (using the axes in Figure 2).

(b) Draw a phase line portrait of the equation $x' = x(mx + b)$ you obtained in (3a). What long term prediction does it imply concerning world population numbers?

(c) Find a formula for solution $x(t)$ of the equation $x' = x(mx + b)$ you obtained in (3a) using $t = 1900$ as an initial condition.

(d) Using your formula for $x(t)$ in (3c) fill in the table below. Plot both $x(t)$ and the data on the same graph for comparison. (Use the same axis as in Figure 1.) model fit the data? How well does the model fit the data?

t	$x =$ world population numbers (billions)	$x(t)$
1900	1.65	
1910	1.75	
1920	1.86	
1930	2.07	
1940	2.30	
1950	2.52	
1960	3.02	
1970	3.70	
1980	4.45	
1990	5.30	
1995	5.68	
2000	6.07	
2005	6.45	

(e) What prediction does the formula in (3c) make concerning world population numbers? Plot the graph of the solution (using a computer) in a window $1900 \leq t \leq 2150$, $0 \leq x \leq 110$.

(4) (a) Find a quadratic fit $q(x) = ax^2 + bx + c$ for the data in Table 2. Do this by the method of nonlinear least squares, i.e., determine numerical values for a, b , and c so as to minimize the sum of the squared residuals: $E = \sum_{i=1900}^{2005} (x_i - q(x_i))^2$ where x_i is the world population number at year i in Table 1. Plot a graph of your computed quadratic (parabola) $q(x)$ together with the data in Table 2. Use the same axes as in Figure 2. Assess this fit (in particular, compare it to the linear fit in (3)). (Hint: you can calculate a, b , and c by setting the three derivatives of E with respect to a, b , and c all equal to 0 and solving the resulting (three linear) equations for a, b , and c .)

(b) Draw the phase line portrait of the differential equation $x' = x(ax^2 + bx + c)$ using the quadratic $ax^2 + bx + c$ you obtained in (4a). What long term prediction does it imply concerning world population numbers?

(5) What is your overall assessment of the three models used to describe and predict world population numbers: the doomsday model in the text, the model in (3) above, and the model in (4) above? What modifications and/or new directions might you suggest for further improvements in modeling world population numbers?