

# ANSWERS PROJECT #1 - Math 250b (Spring 2008)

Section 1, Professor J. M. Cushing

## PART 1

(1) Exercise #23(a-e), page C-29

Suppose a concentration  $z > 0$  of an enzyme is maintained in a container that initially contains a concentration  $c_0 > 0$  of substrate. Suppose no fluid is added or removed from the container.

(a) Write down an initial value problem for the substrate concentration  $c = c(t)$  assuming a Michaelis-Menten uptake rate.

ANSWER:

$$c' = -\frac{mzc}{a+c}, \quad c(0) = c_0$$

(b) Draw the phase line portrait of the differential equation from your answer in (a) and show  $\lim_{t \rightarrow +\infty} c(t) = 0$  when  $c_0 > 0$ .

$$\leftarrow \bullet \rightarrow 0 \leftarrow$$

Here the dot is located at  $c = -a$  but is not an equilibrium. The right side of the differential equation is not defined at this point (so it is excluded from the phase line portrait).

(c) Use a computer program to study the time  $t_{half}$  it takes for the substrate  $c$  to reach half of its initial concentration  $c_0$ . Specifically, formulate a conjecture about how  $t_{half}$  is related to  $c_0$ . What does the halving time  $t_{half}$  do as  $c_0 \rightarrow 0$ ?

ANSWER:  $t_{half}$  appears to increase (linearly)  $c_0$ . As  $c_0 \rightarrow 0$  it appears that  $t_{half}$  approaches a positive limit.

(d) Find a formula for the solution of the initial value problem in (a). You need not find an explicit solution.

ANSWER: The equation in (a) is separable. Separating variables we calculate

$$\begin{aligned} \frac{dc}{dt} &= -\frac{mzc}{a+c} \\ \int_{c_0}^c \frac{a+s}{s} ds &= -mz \int_0^t ds \\ a \ln |s| + s \Big|_{c_0}^c &= -mzt \\ a \ln |c| + c - (a \ln |c_0| + c_0) &= -mzt. \end{aligned}$$

The solution is defined implicitly by the equation  $a \ln c + c + mzt = a \ln c_0 + c_0$ .

(e) Use your solution in (c) to prove or disprove your conjecture in (c).

ANSWER: At  $t_{half}$  the concentration is  $c_0/2$ . By the formula in (d) we have

$$\begin{aligned} a \ln \left( \frac{1}{2} c_0 \right) + \frac{1}{2} c_0 + mzt_{half} &= a \ln c_0 + c_0 \\ mzt_{half} &= a \ln 2 + \frac{1}{2} c_0 \end{aligned}$$

and hence

$$t_{half} = \frac{2a \ln 2 + c_0}{2mz} = \frac{1}{2mz} c_0 + \frac{2a \ln 2}{2mz}.$$

Thus, the formula for  $t_{half}$  as a function of  $c_0$  is linear, with slope  $\frac{1}{2mz}$  and positive intercept  $\frac{2a \ln 2}{2mz}$  when  $c_0 = 0$ . The conjectures in (c) are true.

*Exercise #26(a-b), page C-29*

It has been found in many enzyme reactions that a plot of the reaction rate  $\rho$  against the concentration  $c$  has an inflection point (i.e., a change in concavity) unlike that of the Michaelis-Menten function sketched in Fig. 6.10 of the text book which is concave down for all  $c \geq 0$ . The explanation of phenomenon is that an enzyme can possess several active binding centers for the substrate and the rates of binding activity at one center depends on the activity at other centers. For this reason, such a enzymatic reaction is called “cooperative”, as opposed to one described by the Michaelis-Menten function which is called “noncooperative”. A commonly used expression for such a reaction rate  $\rho$  is “Hill’s Equation”

$$\rho = \frac{mc^n}{a + c^n}z$$

where the exponent  $n > 0$  is determined from data.

(a) Show Hill’s Equation has a positive inflection point as a function of  $c$  if and only if  $n > 1$ . This is why an enzyme/substrate reaction is called “cooperative” if data shows  $n$  is greater than one.

ANSWER:

$$\frac{d\rho}{dc} = mnaz \frac{c^{n-1}}{(a + c^n)^2} > 0$$

so that  $\rho(c)$  is an increasing function of  $c$  with second derivative

$$\frac{d^2\rho}{dc^2} = mnaz \frac{c^{n-2}}{(a + c^n)^3} ((n-1)a - (n+1)c^n)$$

which vanishes if and only if  $c^n = \frac{n-1}{n+1}a$ . There is a positive root of this equation for  $c$  if and only if  $n > 1$ , in which case the inflection point at

$$c = \left( \frac{n-1}{n+1}a \right)^{1/n}$$

because the second derivative  $d^2\rho/dc^2$  changes sign from positive to negative as  $c$  increases through this point.

(b) Determine the phase line portraits of equation  $c' = d(c_{in} - c) - \rho$  when  $\rho$  is taken as Hill’s Equation. Are there any bifurcations for positive values of the parameters?

ANSWER: A unique positive equilibrium  $c_e > 0$  occurs at the intersection point of the straight line with equation  $d(c_{in} - c)$  and the increasing function  $\rho$ . For all positive parameter values the phase portrait is  $\longrightarrow c_e \longleftarrow$  and therefore no bifurcations occur.

**(2) Exercise #2 on page C-26**

Assume the world population is growing exponentially according to the model equations  $x' = rx$ ,  $x(t_0) = x_0$ . Use the data in Table 6.1, with  $t_0 = 1900$  and  $x_0 = 1.65$ , to do the following problems.

**(a)** Determine the value of  $r$  so that the solution interpolates the world population at  $t = 1990$  and obtain a formula for  $x(t)$ .

ANSWER:  $x(t) = 1.65e^{r(t-1900)}$  with  $r \approx 1.2966 \times 10^{-2}$

**(b)** Construct tables of model estimates at each census time and calculate their percent errors.

| Year | World Population<br>(billions) | Interpolation at 1990<br>(billions) | % Error |
|------|--------------------------------|-------------------------------------|---------|
| 1900 | 1.65                           | 1.65                                | 0       |
| 1910 | 1.75                           | 1.88                                | - 7     |
| 1920 | 1.86                           | 2.14                                | - 15    |
| 1930 | 2.07                           | 2.43                                | - 18    |
| 1940 | 2.30                           | 2.77                                | - 20    |
| 1950 | 2.52                           | 3.16                                | - 25    |
| 1960 | 3.02                           | 3.59                                | - 19    |
| 1970 | 3.70                           | 4.09                                | - 11    |
| 1980 | 4.45                           | 4.66                                | - 5     |
| 1990 | 5.30                           | 5.30                                | 0       |

**Exercise #4 on page C-27.**

Obtain the solution of  $x' = kx^{p+1}$ ,  $x(t_0) = x_0$  that interpolates the first three data points at  $t = 1900, 1910$ , and  $1920$  in Table 6.1 of the text book. Calculate the errors and % errors and draw a graph of the solution together with the data. Discuss the results and compare them to the errors in Table 6.3. What doomsday date does this interpolation predict?

|                                                                  | Year | World Population<br>(billions) | Model Prediction<br>(billions) | Error<br>(%) |
|------------------------------------------------------------------|------|--------------------------------|--------------------------------|--------------|
|                                                                  | 1900 | 1.65                           | 1.65                           | 0            |
|                                                                  | 1910 | 1.75                           | 1.75                           | 0            |
|                                                                  | 1920 | 1.86                           | 1.86                           | 0            |
| $x(t) = \frac{1.65}{(7.4939 - 3.4178 \times 10^{-3}t)^{1.6920}}$ | 1930 | 2.07                           | 1.98                           | 4            |
|                                                                  | 1940 | 2.30                           | 2.12                           | 8            |
|                                                                  | 1950 | 2.52                           | 2.27                           | 10           |
|                                                                  | 1960 | 3.02                           | 2.43                           | 19           |
|                                                                  | 1970 | 3.70                           | 2.62                           | 29           |
|                                                                  | 1980 | 4.45                           | 2.83                           | 36           |
|                                                                  | 1990 | 5.30                           | 3.07                           | 42           |

The errors get increasingly worse as the century progresses. The fit is considerably worse than that in Table 3.3. Doomsday occurs at  $t = 2192.6$ .

## PART 2

Table 1 below shows more recent data than that used in the text.

| $t = \text{year}$ | $x = \text{world population numbers (billions)}$ |
|-------------------|--------------------------------------------------|
| 1900              | 1.65                                             |
| 1910              | 1.75                                             |
| 1920              | 1.86                                             |
| 1930              | 2.07                                             |
| 1940              | 2.30                                             |
| 1950              | 2.52                                             |
| 1960              | 3.02                                             |
| 1970              | 3.70                                             |
| 1980              | 4.45                                             |
| 1990              | 5.30                                             |
| 1995              | 5.68                                             |
| 2000              | 6.07                                             |
| 2005              | 6.45                                             |

**TABLE 1**

(1) Plot the data in Table 1 together with the predictions of the doomsday model (6.9) on page C-4 of the text (using the scales in Figure 1). The doomsday model in the text utilizes data from 1900 to 1990. How well does that model predict the world population numbers observed in the subsequent years 1995, 2000 and 2005? What does the subsequent data suggest about the doomsday model?

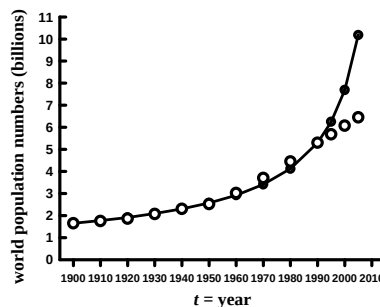


**FIGURE 1**

ANSWER: The doomsday model (6.9) reasonably well fits the data from 1900 to 1990, which was used in the fitting procedure. However, the model does a poor job of predicting the observed data for the subsequent years 1995, 2000 and 2005. The graph shows that the model predictions are greater than the observed values. This suggests that the future predictions of the doomsday model (6.9) (and in particular the date of the “doomsday”) are suspect.

(2) Use the data in Table 1 to approximate the per capita rate of change  $x'(t)/x(t)$  of the world population by calculating the numbers  $\delta(t)$  according to the formula

$$\delta(t) = \frac{1}{x(t)} \frac{x(t + \Delta t) - x(t)}{\Delta t}$$

for years  $t = 1910$  to 2005. Place your answers in a table of the below. Plot the data in Table 2, placing  $x$  on the horizontal axis and  $\delta$  on the vertical axis. (Suggestion: use a vertical scale from 0 to 0.025.)

| $t$  | $x$ (billions) | $\delta$ ( $\times 10^{-3}$ ) |
|------|----------------|-------------------------------|
| 1900 | 1.65           | 6.06                          |
| 1910 | 1.75           | 6.29                          |
| 1920 | 1.86           | 11.29                         |
| 1930 | 2.07           | 11.11                         |
| 1940 | 2.30           | 9.57                          |
| 1950 | 2.52           | 19.84                         |
| 1960 | 3.02           | 22.52                         |
| 1970 | 3.70           | 20.27                         |
| 1980 | 4.45           | 19.10                         |
| 1990 | 5.30           | 14.34                         |
| 1995 | 5.68           | 13.73                         |
| 2000 | 6.07           | 12.53                         |

TABLE 2

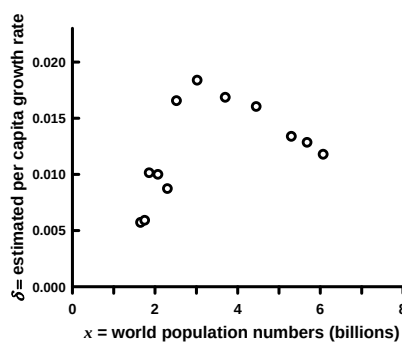


FIGURE 2

- (3) (a) Find a differential equation model for  $x(t)$  of the form

$$\frac{x'}{x} = mx + b$$

by calculating the linear least squares best fit for  $mx + b$  from the data in Table 2. (That is, do a “linear regression” of  $\delta$  on  $x$  to obtain numerical values for  $m$  and  $b$ .) Plot a graph of your calculated line together with the data in Table 2.

ANSWER: The least squares best linear fit to the data in Figure 2 is the straight line  $mx + b$  with  $m \approx 1.180 \times 10^{-3}$ ,  $b \approx 9.915 \times 10^{-3}$ . This line is drawn in Figure 3. The differential equation is therefore

$$x' = x(1.180 \times 10^{-3}x + 9.915 \times 10^{-3}).$$

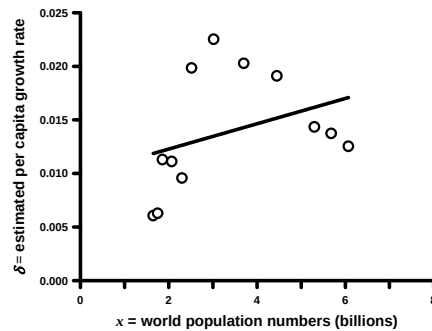


FIGURE 3

- (b) Draw a phase line portrait of the equation  $x' = x(mx + b)$  you obtained in (3a). What long term prediction does it imply concerning world population numbers?

ANSWER: The graph of  $f(x) = x(mx + b) = x(1.180 \times 10^{-3}x + 9.915 \times 10^{-3})$  is an upward cupped parabola with roots  $x \approx -8.403$  and 0. By the graphical method, the phase line portrait is

$$\rightarrow -8.403 \leftarrow 0 \rightarrow$$

For  $x(0) > 0$  this phase line portrait shows unbounded population growth in the future.

- (c) Find a formula for solution of the equation  $x' = x(mx + b)$  you obtained in (3a) using  $t = 1900$  as an initial condition.

The equation is autonomous and therefore separable.

ANSWER:

$$\begin{aligned} \frac{dx}{dt} &= x(mx + b) \\ \int_{x_0}^x \frac{1}{x(mx + b)} dx &= \int_{t_0}^t dt \\ \int_{x_0}^x \frac{1}{b} \left( \frac{1}{x} - \frac{m}{b + mx} \right) dx &= t - t_0 \\ \ln \frac{|x|}{|b + mx|} &= b(t - t_0) \end{aligned}$$

This defines the solution  $x$  implicitly. Solving for  $x$ , we obtain the explicit solution formula

$$x(t) = bx_0 \frac{e^{b(t-t_0)}}{b + mx_0 - x_0 m e^{b(t-t_0)}}$$

Now we let  $m \approx 1.180 \times 10^{-3}$ ,  $b \approx 9.915 \times 10^{-3}$  and  $t_0 = 1900$ ,  $x_0 = 1.65$  to get

$$x(t) = 1.636 \frac{\exp(9.915 \times 10^{-3}(t - 1900))}{1.1862 - 0.1947 \exp(9.915 \times 10^{-3}(t - 1900))}.$$

(d) Using your formula for  $x(t)$  in (3c) fill in the table below. Plot both  $x(t)$  and the data on the same graph for comparison. How well does the model fit the data?

| $t$  | world population numbers (billions) | $x(t) \approx$ |
|------|-------------------------------------|----------------|
| 1900 | 1.65                                | 1.65           |
| 1910 | 1.75                                | 1.86           |
| 1920 | 1.86                                | 2.10           |
| 1930 | 2.07                                | 2.38           |
| 1940 | 2.30                                | 2.71           |
| 1950 | 2.52                                | 3.10           |
| 1960 | 3.02                                | 3.56           |
| 1970 | 3.70                                | 4.11           |
| 1980 | 4.45                                | 4.78           |
| 1990 | 5.30                                | 5.62           |
| 1995 | 5.68                                | 6.11           |
| 2000 | 6.07                                | 6.67           |
| 2005 | 6.45                                | 7.30           |

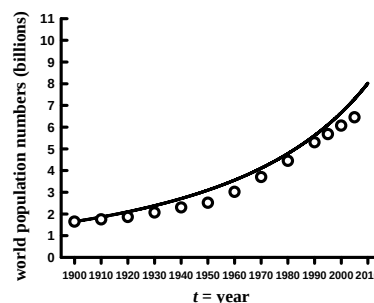


FIGURE 4

This model has reasonable small errors throughout the century, but it consistently over predicts the actual data.

(e) What prediction does the formula in (3c) make concerning world population numbers? Plot (using a computer) the graph of the solution.

ANSWER: The model predicts a doomsday when denominator vanishes, i.e., when

$$1.1862 - 0.1947 \exp(9.915 \times 10^{-3}(t - 1900)) = 0.$$

Solving this equation for  $t$ , we get a doomsday date of approximately 2082.3. The slope field and solution graph for the initial value problem

$$x' = x(1.180 \times 10^{-3}x + 9.915 \times 10^{-3}), \quad x(1900) = 1.675$$

appears in Figure 5.

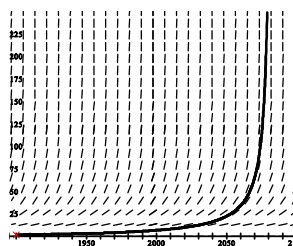


FIGURE 5

(4) (a) Find a quadratic fit  $q(x) = ax^2 + bx + c$  for the data in Table 2. Do this by the method of nonlinear least squares, i.e., determine numerical values for  $a, b$ , and  $c$  so as to minimize the sum of the squared residuals:  $E = \sum_{i=1900}^{2005} (x_i - q(x_i))^2$  where  $x_i$  is the world population number at year  $i$  in Table 1. Plot a graph of your computed quadratic (parabola)  $q(x)$  together with the data in Table 2. Use the same axes as in Figure 2. Assess this fit (in particular, compare it to the linear fit in (3)). (Hint: you can calculate  $a, b$ , and  $c$  by setting the three derivatives of  $E$  with respect to  $a, b$ , and  $c$  all equal to 0 and solving the resulting (three linear) equations for  $a, b$ , and  $c$ .)

ANSWER: The quadratic least square fit is  $ax^2 + bx + c$  with

$$\begin{aligned} a &= -2.61 \times 10^{-3} \\ b &= 20.89 \times 10^{-3} \\ c &= -20.52 \times 10^{-3} \end{aligned}$$

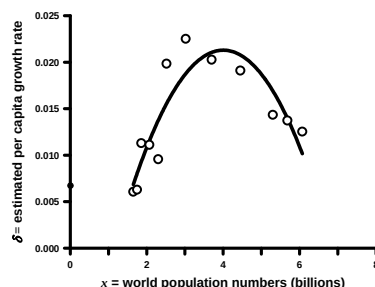


FIGURE 6

(b) Draw the phase line portrait of the differential equation  $x' = x(ax^2 + bx + c)$  using the quadratic  $ax^2 + bx + c$  you obtained in (4a). What long term prediction does it imply concerning world population numbers?

ANSWER: The differential equation is  $x' = x(ax^2 + bx + c)$  with coefficients from (a), i.e.,

$$x' = x(-2.61x^2 + 20.89x - 20.52) \times 10^{-3}.$$

The equilibria are  $x = 0$  and roots of  $-2.61x^2 + 20.89x - 20.52$ , which are approximately  $x \approx 1.15$  and  $6.86$ . The function  $f(x) = x(ax^2 + bx + c)$  is a cubic that has these three roots and that is negative for large  $x$ . The, by the geometric method, we obtain the phase line portrait

$$\rightarrow 0 \leftarrow 1.15 \rightarrow 6.86 \leftarrow$$

By this portrait we see that the initial condition 1.65 yields a solution that increases to the (approximate) equilibrium  $x = 6.86$ . This is no longer a doomsday prediction. World population is predicted to equilibrate at 6.86 billion.

(5) What is your overall assessment of the three models used to describe and predict world population numbers (namely, the doomsday model in the text, the model from (3), and the model from (4))? What modifications and/or new directions might you suggest for further improvements in modeling world population numbers?

ANSWER: The graph of the solution of the initial value problem

$$x' = x(-2.61x^2 + 20.89x - 20.52) \times 10^{-3}, \quad x(1900) = 1.65$$

appears in Figure 7. This model fits the data reasonably well at the beginning and end of the century, although it over-predicts during the middle of the century. Its errors are commensurate with those of the model in (3), but it does better than that model at the end of the century. This model also predicts an finite equilibrium for the world population and not a doomsday as predicted by the model in the text and by the model in (3). For these reasons, perhaps this model should be considered the best of the three models.

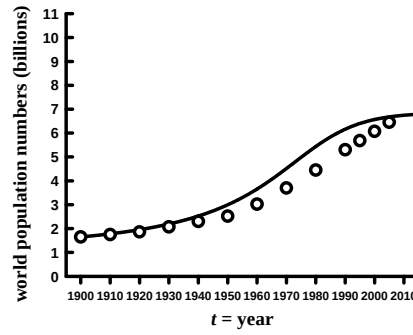


FIGURE 7