

Answers To Test 2

Math 250a, Section 2

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Professor J. M. Cushing

NAME (please print) : _____

Instructions. Do all problems. **Show your work and justify answers** in order to receive full credit. Use of textbook, calculators, and notes is allowed. Total points = 100.

1. (10 points) Evaluate the following integral. **Show your work.**

$$\int (\sin 2\theta)^6 \cos 2\theta d\theta = \frac{1}{14} (\sin 2\theta)^7 + c$$

Let $w = \sin 2\theta$ so that $dw = 2 \cos 2\theta$. Then

$$\int (\sin 2\theta)^6 \cos 2\theta d\theta = \int w^6 \frac{1}{2} dw = \frac{1}{14} w^7 + c.$$

2. (10 points) Evaluate the following integral. **Show your work.**

$$\int \frac{1}{(x+2)(x+3)} dx = \ln \frac{|x+2|}{|x+3|} + c$$

By partial fraction decomposition $\frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}$ and hence

$$\begin{aligned} \int \frac{1}{(x+2)(x+3)} dx &= \int \frac{1}{x+2} dx - \int \frac{1}{x+3} dx \\ &= \ln|x+2| - \ln|x+3| + c. \end{aligned}$$

3. (15 points) Find the *exact* value of the integral

$$\int_1^{3/2} \frac{z-1}{\sqrt{2z-z^2}} dz = 1 - \frac{1}{2} \sqrt{3}$$

One method: Let $w = 2z - z^2$ so that $dw = 2(1-z)dz$ and

$$\int_{z=1}^{z=3/2} \frac{z-1}{\sqrt{2z-z^2}} dz = -\frac{1}{2} \int_{w=1}^{w=3/4} \frac{1}{\sqrt{w}} dw = -\sqrt{w} \Big|_1^{3/4}.$$

Another method: Completing the square under the square root and letting $z-1 = \sin \theta$ (so that $dz = \cos \theta d\theta$) we have

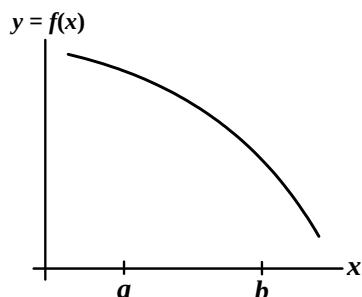
$$\begin{aligned} \int_1^{3/2} \frac{z-1}{\sqrt{2z-z^2}} dz &= \int_1^{3/2} \frac{z-1}{\sqrt{1-(z-1)^2}} dz \\ &= \int_{\theta=0}^{\theta=\pi/6} \frac{\sin \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta = \int_0^{\pi/6} \frac{\sin \theta}{\sqrt{\cos^2 \theta}} \cos \theta d\theta \end{aligned}$$

Since $\cos \theta > 0$ on the interval of integration, it follows that $\sqrt{\cos^2 \theta} = \cos \theta$ and hence

$$\int_1^{3/2} \frac{z-1}{\sqrt{2z-z^2}} dz = \int_{\theta=0}^{\theta=\pi/6} \sin \theta d\theta = -\cos \theta \Big|_0^{\pi/6} = -\cos \frac{\pi}{6} - (-\cos 0) = -\frac{\sqrt{3}}{2} + 1.$$

4. (15 points) For the function whose graph appears in the figure below, rank the following from lowest to highest. **Justify your answers**

$MID(5)$, $TRAP(5)$, $RIGHT(5)$, $LEFT(5)$, $MID(50)$, $TRAP(50)$ and Exact answer $= \int_a^b f(x)dx$



ANSWER:

$$RIGHT(5) < TRAP(5) < \int_a^b f(x)dx < MID(50) < MID(5) < LEFT(5)$$

Graph is decreasing and concave down. Therefore, we have

$$RIGHT(5) < TRAP(5) < \int_a^b f(x)dx < MID(5) < LEFT(5).$$

By increasing the number of mesh points, $TRAP$ increasingly and MID decreasingly approach the exact answer.

5. (15 points) Let $f(s)\Delta s$ denote probability that a particle travels between s and $s + \Delta s$ meters from a source of pollution. ($f(s)$ is called a probability density and has units of 1/meters.) The expected distance travelled by an typical particle emitted from this source is given by the formula $\int_0^{+\infty} sf(s)ds$. Suppose the probability decreases exponentially with distance from the source according to the formula $f(s) = ae^{-as}$, where a is a positive constant.
- (a) What are the units for a ?

ANSWER: 1/meters

- (b) What is the expected distance travelled by a particle? **Show your work.**

ANSWER: $s = 1/a$ meters

Using integration by parts and

$$\int_0^{+\infty} sf(s)ds = \int_0^{+\infty} s(ae^{-as})ds = \lim_{b \rightarrow +\infty} \int_0^b s(ae^{-as})ds$$

we calculate

$$\int_0^b s(ae^{-as})ds = -se^{-as} \Big|_0^b + \int_0^b e^{-as}ds = (-be^{-ab} + 0) + \left(-\frac{1}{a}e^{-as}\right) \Big|_0^b = -be^{-ab} - \frac{1}{a}e^{-ab} + \frac{1}{a}.$$

and then

$$\lim_{b \rightarrow +\infty} \int_0^b s(ae^{-as})ds = \lim_{b \rightarrow +\infty} \left[-be^{-ab} - \frac{1}{a}e^{-ab} + \frac{1}{a} \right] = -0 - 0 + \frac{1}{a}.$$

6. (15 points) Consider the improper integral

$$\int_1^2 \frac{1}{(x-1)\ln x} dx.$$

- (a) By investigating the behavior of $\ln x$ near $x = 1$, make a conjecture about the convergence or divergence of this integral. (HINT: as discussed in class, approximate $y = \ln x$ by its tangent line at $x = 1$.) **Justify your answer.**

ANSWER: Divergent

$\ln x \approx x - 1$ and hence $\frac{1}{(x-1)\ln x} \approx \frac{1}{(x-1)^2}$ for $x \approx 1$. The improper integral $\int_1^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{w^2} dw$ diverges, so we conjecture $\int_1^2 \frac{1}{(x-1)\ln x} dx$ diverges.

- (b) Use the Comparison Test to prove your conjecture in (a). **Show your work.**

Since we conjectured divergence in (a), we look for a *smaller* comparison function. Since $y = \ln x$ is concave down, its graph lies below that of its tangent, that is to say, $\ln x \leq x - 1$. Therefore, for x between 1 and 2 we have

$$\frac{1}{(x-1)\ln x} \geq \frac{1}{(x-1)^2} \geq 0$$

Since $\int_1^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{w^2} dw$ diverges, the Comparison Test implies $\int_1^2 \frac{1}{(x-1)\ln x} dx$ diverges.

7. (20 points) Use the method of slicing to obtain the formula for the volume of a (right circular) cone of height h whose base has radius r . **Show your work.**

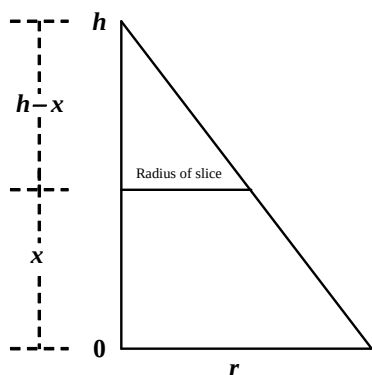
ANSWER: $\frac{1}{3}\pi r^2 h$

Slices, taken at position x between 0 and h and perpendicular to the axis of the cone, are approximately disks with thickness Δx and a radius that depends on the position x of the slice. From the similar triangles in the figure below we see that this radius is $\frac{r}{h}(h-x)$ and hence the volume of the slice $\Delta V \approx \pi \left(\frac{r}{h}(h-x)\right)^2 \Delta x$. In the limit we obtain the exact volume:

$$V = \lim_{\Delta x \rightarrow 0} \sum \pi \left(\frac{r}{h}(h-x)\right)^2 \Delta x = \int_0^h \pi \left(\frac{r}{h}(h-x)\right)^2 dx.$$

Let $w = \frac{r}{h}(h-x)$ so that $dw = -\frac{r}{h}dx$ and

$$V = \pi \int_r^0 w^2 \left(-\frac{h}{r}dw\right) = \pi \frac{h}{r} \int_0^r w^2 dw = \pi \frac{h}{r} \frac{1}{3} w^3 \Big|_0^r = \frac{1}{3} \pi \frac{h}{r} r^3 = \frac{1}{3} \pi r^2 h.$$



8. **EXTRA CREDIT** (10 points) A pipe running underneath the beach makes a connection between the ocean and a (man-made) tide pool. Typically, the ocean-end of the pipe is above the water line. However, at high tide waves crashing on the shore will cause water to flow through the pipe from the ocean into the pool. Suppose the volume velocity (that is, the rate of number of liters (L) of water flowing through the pipe from the beach to the tide pool) for a single wave crashing on the shore is given by $\varphi(t) = e^{-t/\tau} \sin(\alpha t)$ where t is in seconds (s). Let the constants τ and α have values of 4s and 2/s respectively.

- (a) Calculate the total amount of water flowing through the pipe due to one wave crashing on the shore. Express your answer in mL .

ANSWER: _____

From the Table of Integrals

$$\begin{aligned} \int_0^{+\infty} e^{-t/\tau} \sin(\alpha t) dt &= \lim_{b \rightarrow \infty} \int_0^b e^{-t/\tau} \sin(\alpha t) dt \\ &= \lim_{b \rightarrow \infty} \frac{1}{\left(\frac{1}{\tau}\right)^2 + \alpha^2} e^{-t/\tau} \left(\frac{1}{\tau} \sin(\alpha t) - \alpha \cos(\alpha t) \right) \Bigg|_0^b \\ &= \frac{1}{\left(\frac{1}{\tau}\right)^2 + \alpha^2} \lim_{b \rightarrow \infty} \left[e^{-b/\tau} \left(\frac{1}{\tau} \sin(\alpha b) - \alpha \cos(\alpha b) \right) - [0 - \alpha] \right] \\ &= \frac{\tau^2}{1 + \tau^2 \alpha^2} \alpha. \end{aligned}$$

Using $\tau = 4$ and $\alpha = 2$ we have

$$\int_0^{+\infty} e^{-t/\tau} \sin(\alpha t) dt = \frac{32}{65} L = \frac{32}{65} L \times 1000 \frac{mL}{L} \approx 0.4923 \text{ mL}$$

- (b) Is the net flow of water going from the ocean to the pool or vice-versa? Why?

Since the answer in (a) is positive, the net flow is into the pool.