

Answers To Test 1

Math 250a, Section 2

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NAME (please print) : _____

Instructions. Do all problems. **Show your work and justify answers** in order to receive full credit. Use of textbook, calculators, and notes is allowed. Total points = 100.

1. (30 points) Find the derivative of the functions below. a is a constant. **Show your work.**

(a) $f(x) = 2^{\sin x} \cos x$

ANSWER: Using the product rule and $2^{\sin x} = e^{(\ln 2) \sin x}$, we obtain

$$\begin{aligned} \frac{d(2^{\sin x} \cos x)}{dx} &= \frac{d 2^{\sin x}}{dx} \cos x + 2^{\sin x} \frac{d \cos x}{dx} = ((\ln 2) 2^{\sin x} \cos x) \cos x - 2^{\sin x} \sin x \\ &= 2^{\sin x} ((\ln 2) \cos^2 x - \sin x). \end{aligned}$$

(b) $f(x) = \frac{a^2 - x^2}{a^2 + x^2}$

ANSWER: Using the quotient rule, we find

$$\frac{df(x)}{dx} = \frac{(a^2 + x^2)(-2x) - (a^2 - x^2)2x}{(a^2 + x^2)^2} = \frac{-4a^2x}{(a^2 + x^2)^2}$$

(c) $r(t) = \ln(\sin(\frac{t}{a}))$

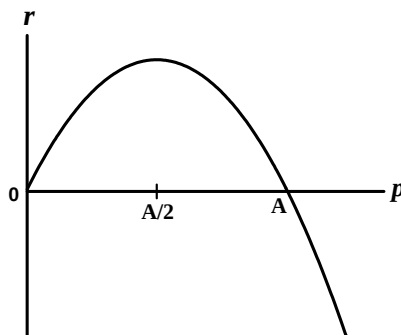
ANSWER: Using the chain rule (twice) we have

$$\frac{dr(t)}{dt} = \frac{1}{\sin(\frac{t}{a})} \frac{d \sin(\frac{t}{a})}{dt} = \frac{1}{\sin(\frac{t}{a})} \cos\left(\frac{t}{a}\right) \frac{1}{a} = \frac{1}{a} \cot\left(\frac{t}{a}\right).$$

2. (15 points) Suppose A is the amount of a chemical that is subjected to a catalyzed reaction whose product is itself a catalyst. (Such a reaction is said to be autocatalytic.) Assume the rate, r , of an autocatalytic reaction is proportional to the quantity p of the catalytic product produced multiplied by the quantity of the original chemical remaining.

(a) Express r as a function of p and sketch a graph of this function.

ANSWER: $r(p) = kp(A - p)$ where k is a constant of proportionality. Since $p < A$ the constant k must be positive so that the reaction rate is positive. The graph of this quadratic polynomial $r(p) = -kp^2 + kAp$ is a concave downward parabola (because of the negative coefficient $-k$ on p^2) with roots at $p = 0$ and A .



(b) What is the value of p when the reaction is proceeding fastest? **Justify your answer and show your work.**

ANSWER: The maximum of the parabolic graph of $r(p)$ occurs at the critical point p where $r'(p) = 0$, i.e., where $-2kp + kA = 0$. Thus, the maximum occurs at $p = A/2$.

(c) If initially small, what happens to the amount of the catalytic product p as time increases indefinitely?

ANSWER: Since the reaction rate r is positive for small p , it follows that p will increase. It will continue to increase until the reaction rate drops to 0, which occurs when $p = A$. Thus, p approaches A as time increases indefinitely.

3. (15 points) A news broadcast in early 1993 said the typical American's annual income was changing at a rate of $r(t) = 40(1.002)^t$ dollars per month, where t is in months from January 1, 1993. How much did the typical American's income change during 1993?

Show your work.

ANSWER: Let $R(t)$ denote the annual income at time t . Then that $r(t) = \frac{dR(t)}{dt}$. The change in income during 1993 is $R(12) - R(0)$, which by the Fundamental Theorem of Calculus, is $R(12) - R(0) = \int_0^{12} r(t)dt$. We calculate the anti-derivative $R(t)$ of $r(t)$ to be

$$R(t) = \int r(t)dt = \int 40(1.002)^t dt = \int 40e^{(\ln 1.002)t} dt = 40 \frac{1}{\ln 1.002} e^{(\ln 1.002)t}$$

we have that

$$\int_0^{12} r(t)dt = 40 \frac{1}{\ln 1.002} e^{(\ln 1.002)t} \Big|_{t=0}^{12} = 40 \frac{1}{\ln 1.002} (e^{(\ln 1.002)12} - 1) \approx \$485.80.$$

4. (15 points) Consider the three data pairs $(x,y) = (1,2), (2,3), (3,5)$.

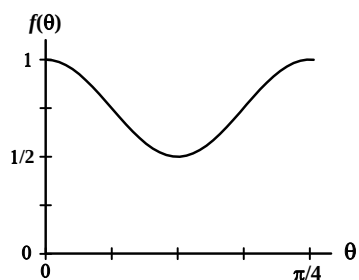
- (a) Write down the total error E (in terms of squared residuals) for a straight line that passes through (interpolates) the origin $(x,y) = (0,0)$.

ANSWER: A straight line through the origin has the form $y = mx$. Then the total error is the sum of the squared residuals: $E = (2 - m)^2 + (3 - 2m)^2 + (5 - 3m)^2$.

- (b) Subject to passing through (interpolating) the origin, as in (a), use calculus to find the “linear least squares” best fit the three data points. **Show your work.**

ANSWER: The total error $E = 14m^2 - 46m + 38$ is a quadratic polynomial in m and therefore has a unique global minimum at the critical point where $E' = 28m - 46 = 0$, i.e., at $m = 23/14 \approx 1.643$. The “least squares” best linear fit is therefore $y = (23/14)x$.

5. (15 points) Consider a function $f(\theta)$ whose graph is



- (a) Write $f(\theta)$ as a cosine function. **Justify your answer.**

ANSWER: For the general cosine function $f(\theta) = D + A \cos(\omega(\theta - \varphi))$ we see from the graph that the average is $D = 3/4$ and the amplitude is $A = \frac{1}{2}(1 - \frac{3}{4}) = \frac{1}{4}$. Since the maximum occurs at $\theta = 0$ the phase shift is $\varphi = 0$. The period $\frac{2\pi}{\omega}$ is $\frac{\pi}{4}$ and hence $\omega = 8$.

$$f(\theta) = \frac{3}{4} + \frac{1}{4} \cos 8\theta.$$

- (b) Calculate the area under the curve. **Show your work.**

ANSWER: The area is the integral of $f(\theta)$ over the interval $0 \leq \theta \leq \pi/4$, which we can evaluate by using the Fundamental Theorem of Calculus. An anti-derivative of $f(\theta)$ is $F(\theta) = \frac{3}{4}\theta + \frac{1}{32} \sin 8\theta$ and so

$$\int_0^{\pi/4} f(\theta) d\theta = F\left(\frac{\pi}{4}\right) - F(0) = \left(\frac{3}{4} \frac{\pi}{4} + \frac{1}{32} \sin 8 \frac{\pi}{4}\right) - 0 = \left(\frac{3\pi}{16} + 0\right) - 0 = \frac{3\pi}{16}.$$

(You might also notice from the graph that the area under the curve equals that of a rectangle with height $3/4$. This is because the area above the line at height $3/4$ equals that below the line. The area of the rectangle is the base times the height or $3/4 \times \pi/4 = 3\pi/16$.)

- (c) Write $f(\theta)$ as a sine function. **Justify your answer.**

ANSWER: For the general sine function $f(\theta) = D + A \sin(\omega(\theta - \varphi))$ we see from the graph (as in part (a)) that the average is $D = 3/4$ and the amplitude is $A = \frac{1}{4}$. As in (a) the frequency is $\omega = 8$. Since the maximum occurs at $\theta = 0$ the phase shift is one quarter of the period, i.e., $\varphi = -\frac{1}{4} \frac{\pi}{4} = -\frac{\pi}{16}$.

$$f(\theta) = \frac{3}{4} + \frac{1}{4} \sin 8\left(\theta + \frac{\pi}{16}\right).$$

6. (10 points) The acidity of a solution is typically described by its pH value, which is defined as

$$pH = -\log[H^+]$$

where $[H^+]$ is the molar concentration of free hydrogen ions. pH values typically range from 0 to 14, where a smaller pH value is considered acidic and a larger value basic. Note: $1\text{mL} = 1 \times 10^{-3}\text{L}$.

- (a) Lemon juice typically has an ion concentration of $5.4 \times 10^{-3}\text{M}$, where M stands for moles per liter (mol/L). Suppose you dilute 10 mL of lemon juice with 90 mL of water (which has a hydrogen ion concentration of $1 \times 10^{-7}\text{M}$). What is $[H^+]$ of the diluted mix? Comment on the water's relative contribution of $[H^+]$ ions.

ANSWER: $1/10$ that of lemon juice, or $[H^+] = \frac{1}{10} \times 5.4 \times 10^{-3} = 5.4 \times 10^{-4}\text{M}$.

- (c) How much has the pH changed between the pure lemon juice and the dilution? Has it increased or decreased?

ANSWER: The pH of lemon juice is $-\log(5.4 \times 10^{-3}) \approx 2.3$ and the pH of the dilution is $-\log(5.4 \times 10^{-4}) \approx 3.3$, so it increased.

- (d) What if we diluted the 10 mL of lemon juice with 990 mL of water. Approximately how much would you expect the pH to change? Would it increase or decrease?

ANSWER: pH would be ≈ 4.3 , an increase.

- (f) Suppose you could remove hydrogen ions from a solution. As the number ions present got very small, what happens to the pH value? Is there an asymptote if you plotted pH versus $1/[H^+]$?

ANSWER: pH would keep increasing without bound, so there is no asymptote