

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, November 29

<http://math.arizona.edu/~cushing/250a.html>

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

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$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \text{ is a matrix such that } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



called the *multiplicative inverse*

Questions

- Does every matrix have an inverse ?
- If a matrix has an inverse, how do we calculate it ?
- If not, does this mean the problem have no solution ?

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Not every number has an inverse (0 doesn't).

So, by analogy we suspect *not all matrices have an inverse*.

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Theorem

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ has an inverse if and only if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0$$

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$$a_{11}a_{22} - a_{21}a_{12} \neq 0$$

This number is called the *determinant* of the matrix

Examples

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} : \text{determinant} = 0 \times 0 - 0 \times 0 = 0$$

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$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

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$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = 1.2 \times (-1) - 0.8 \times 1.5 = -2.4 \Rightarrow \text{has an inverse}$$

Questions

- If a matrix is invertible, how do calculate its inverse ?

A Method

If $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} =$$

Questions

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If $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

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interchange diagonal entries

Questions

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change signs of off-diagonal entries

Questions

- If a matrix is invertible, how do calculate its inverse ?

A Method

If $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{a_{22}}{a_{11}a_{22} - a_{21}a_{12}} & -\frac{a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \\ -\frac{a_{21}}{a_{11}a_{22} - a_{21}a_{12}} & \frac{a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \end{pmatrix}$$

divide by the determinant

Questions

- If a matrix is invertible, how do calculate its inverse ?

A Method

If $a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$



multiplication (or division) of a matrix by a number
(done term-wise)

Examples

$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = 1.2 \times (-1) - 0.8 \times 1.5 = -2.4 \Rightarrow \text{has an inverse}$$

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$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \\ & 1.2 \end{pmatrix}$$

interchange diagonal entries

Examples

$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = 1.2 \times (-1) - 0.8 \times 1.5 = -2.4 \Rightarrow \text{has an inverse}$$

$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & -1.5 \\ -0.8 & 1.2 \end{pmatrix}$$

change signs of off-diagonal entries

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$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix}^{-1} = \frac{1}{-2.4} \begin{pmatrix} -1 & -1.5 \\ -0.8 & 1.2 \end{pmatrix}$$

divide by the determinant

Examples

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$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix}^{-1} = \frac{1}{-2.4} \begin{pmatrix} -1 & -1.5 \\ -0.8 & 1.2 \end{pmatrix} \approx \begin{pmatrix} 0.4167 & 0.6250 \\ 0.333 & -0.5000 \end{pmatrix}$$

Examples

Solve for x and y :

$$1.2x + 1.5y = 1$$

$$0.8x - y = -2$$

Matrix formulation :

$$\begin{pmatrix} 1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

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Note: $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{33} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any matrix.

Examples

Solve for x and y :

$$\boxed{-1.2x - 1.5y = 0}$$

$$0.8x - y = 0$$



Make a change.

Matrix formulation :

$$\begin{pmatrix} \boxed{-1.2} & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

What does this mean ? There is no solution ?

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Look what happens in our 1 equation analogy :

$$a x = 0$$

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when a equals 0 (i.e., when a is not invertible)

$$0x = 0$$

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Solve for x and y :

$$-1.2x + 1.5y = 0$$

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Look what happens in our 1 equation analogy :

$$a x = 0$$

when a equals 0 (i.e., when a is not invertible)

$$0x = 0$$

all values of x are solutions !

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

Similarly in this matrix equation
-- there are infinitely many solutions !

Why ?

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

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Matrix solution :


$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

Multiple first equation by

$$-\frac{0.8}{1.2} = -\frac{2}{3}$$

Examples

Solve for x and y :

$$\begin{aligned} -1.2x + 1.5y &= 0 \\ 0.8x - y &= 0 \end{aligned}$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix}$: determinant $= -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow$ non-invertible

Multiple first equation by

$$-\frac{0.8}{1.2} = -\frac{2}{3}$$

$$\begin{aligned} 0.8x - y &= 0 \\ 0.8x - y &= 0 \end{aligned}$$

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

$$0.8x - y = 0$$

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To solve this system, we need only solve *one equation*!

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

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$$0.8x - y = 0$$

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To solve this system, we need only solve *one equation*!

Choose x to be anything you want & let the equation determine y .

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

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For example, if $x = 1$ then $y = 0.8$.

Examples

Solve for x and y :

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$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

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For example, if $x = 1$ then $y = 0.8$.

For example, if $x = 2$ then $y = 1.6$.

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

$$0.8x - y = 0$$

$$0.8x - y = 0$$

For example, if $x = 1$ then $y = 0.8$.

For example, if $x = 2$ then $y = 1.6$.

For example, if $x = -1/2$ then $y = -0.4$.

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

$$0.8x - y = 0$$

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If $x = c$, then $y = 0.8c$.

Examples

Solve for x and y :

$$-1.2x + 1.5y = 0$$

$$0.8x - y = 0$$

Matrix formulation :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Matrix solution :

$$\begin{pmatrix} -1.2 & 1.5 \\ 0.8 & -1 \end{pmatrix} : \text{determinant} = -1.2 \times (-1) - 0.8 \times 1.5 = 0 \Rightarrow \text{non-invertible}$$

$$0.8x - y = 0$$

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$$\text{Infinitely many solutions: } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ 0.8c \end{pmatrix} = c \begin{pmatrix} 1 \\ 0.8 \end{pmatrix}$$

where c is an arbitrary constant

Homogeneous system of equations

$$a_{11}x + a_{12}y = 0$$

$$a_{21}x + a_{22}y = 0$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Homogeneous system of equations

$$\begin{aligned} a_{11}x + a_{12}y &= 0 \\ a_{21}x + a_{22}y &= 0 \end{aligned} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $a_{11}a_{22} - a_{21}a_{12} = 0$ (the matrix is *not* invertible),
then the equations are *multiples of each other*.

Homogeneous system of equations

$$\begin{aligned} a_{11}x + a_{12}y &= 0 \\ a_{21}x + a_{22}y &= 0 \end{aligned} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $a_{11}a_{22} - a_{21}a_{12} = 0$ (the matrix is *not* invertible),
then the equations are *multiples of each other*.

Solve one equation, you automatically solve the other.

Homogeneous system of equations

$$\begin{aligned} a_{11}x + a_{12}y &= 0 \\ a_{21}x + a_{22}y &= 0 \end{aligned} \qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If $a_{11}a_{22} - a_{21}a_{12} = 0$ (the matrix is *not* invertible),
then the equations are *multiples of each other*.

Solve one equation, you automatically solve the other.

Choose one variable to be anything you want
and let an equation give you the other.

Punch lines

Homogeneous system of equations

$$\begin{aligned} a_{11}x + a_{12}y &= 0 \\ a_{21}x + a_{22}y &= 0 \end{aligned} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- If $a_{11}a_{22} - a_{21}a_{12} \neq 0$,
equations have only one solution: $x = 0, y = 0$.
- If $a_{11}a_{22} - a_{21}a_{12} = 0$,
equations have infinitely many solutions.

Return to our recursion problem :

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\lim_{n \rightarrow +\infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Motivated by the solution formula $x_n = r^n x_0$ for one recursion equation, let's look for exponential solutions.

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Question: Can we find constants r and c_1, c_2 so that

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \quad \text{is a solution ?}$$

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i.e., so that
$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \quad \text{for all } n = 0, 1, 2, 3, \dots$$

Example

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

Example

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

Is a solution if
(and only if)

$$\begin{aligned}c_1 r^{n+1} &= 0.2c_1 r^n + 1.5c_2 r^n \\c_2 r^{n+1} &= 0.8c_1 r^n + 0.4c_2 r^n\end{aligned}$$

for all n .

Example

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Cancel r^n

$$\begin{aligned}c_1 r &= 0.2c_1 + 1.5c_2 \\c_2 r &= 0.8c_1 + 0.4c_2\end{aligned}$$

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Cancel r^n

$$\begin{aligned}c_1 r &= 0.2c_1 + 1.5c_2 \\c_2 r &= 0.8c_1 + 0.4c_2\end{aligned}$$

Gather terms

$$\begin{aligned}(0.2 - r)c_1 + 1.5c_2 &= 0 \\0.8c_1 + (0.4 - r)c_2 &= 0\end{aligned}$$

Example

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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Have only the solution $c_1 = 0, c_2 = 0$
unless the determinant equals 0.

Example

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$$(0.2 - r)c_1 + 1.5c_2 = 0$$

$$0.8c_1 + (0.4 - r)c_2 = 0$$

$$\text{Determinant: } (0.2 - r)(0.4 - r) - 1.5 \times 0.8 = 0$$

Example

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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Determinant: $r^2 - 0.6r - 1.12 = 0$

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One choice is $r = 1.4$

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$$-1.2c_1 + 1.5c_2 = 0$$

$$0.8c_1 - c_2 = 0$$

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$$-1.2c_1 + 1.5c_2 = 0$$

$$0.8c_1 - c_2 = 0$$

Since the determinant equals 0,
these equations are multiples of each other.

We need solve only one.

Example

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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One choice is $r = 1.4$

~~$$-1.2c_1 + 1.5c_2 = 0$$~~

$$0.8c_1 - c_2 = 0$$

Pick $c_1 = c$, then $c_2 = 0.8c$

Example

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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$$0.8c_1 - c_2 = 0$$

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$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

We get infinitely many exponential solutions :

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

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We get infinitely many exponential solutions :

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

An example: $c = 5$ gives $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 5 \times 1.4^n \\ 4 \times 1.4^n \end{pmatrix}$

Example

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$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 5 \times 1.4^n \\ 4 \times 1.4^n \end{pmatrix} \quad \text{Since} \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

this is the same sequence we started to calculate recursively
in an earlier example !

Example

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this is the same sequence we started to calculate recursively
in an earlier example !

$$\lim_{n \rightarrow +\infty} x_n = \infty, \quad \lim_{n \rightarrow +\infty} y_n = \infty \quad (\text{as we suspected})$$

Both juvenile & adult numbers are exponentially unbounded .

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

r and c_1, c_2
must satisfy

$$\begin{pmatrix} c_1 r^{n+1} \\ c_2 r^{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

for all n .

Eigenvalue / Eigenvector Method

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for all n .

$$c_1 r^{n+1} = a_{11} c_1 r^n + a_{12} c_2 r^n$$

$$c_2 r^{n+1} = a_{21} c_1 r^n + a_{22} c_2 r^n$$

Eigenvalue / Eigenvector Method

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Cancel r^n

$$c_1 r = a_{11} c_1 + a_{12} c_2$$

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Eigenvalue / Eigenvector Method

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Cancel r^n

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

and gather terms

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Eigenvalue / Eigenvector Method

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$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Two equations, two unknowns c_1, c_2 .

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Equations have only the solution $c_1 = 0, c_2 = 0$,
unless the determinant equals 0 :

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Cancel r^n

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

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Equations have only the solution $c_1 = 0, c_2 = 0$,
unless the determinant equals 0 :

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

Pick r as a root of this equation --- then solve for c_1, c_2

Eigenvalue / Eigenvector Method

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$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \quad \text{is a solution if } r \text{ satisfies the quadratic equation}$$

$$(a_{11} - r)(a_{22} - r) - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

A quadratic equation (the *characteristic equation*)

Eigenvalue / Eigenvector Method

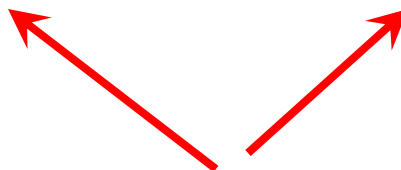
$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$ is a solution if r satisfies the quadratic equation

$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$



Recall: we need solve only one of these two equations.

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix} \text{ is a solution if } r \text{ satisfies the quadratic equation}$$
$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

r is called an *eigenvalue* of the matrix

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ is called an } \textit{eigenvector} \text{ associated with } r$$

Eigenvalue / Eigenvector Method

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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$$r^2 - (a_{11} + a_{22})r + a_{11}a_{22} - a_{21}a_{12} = 0$$

and c_1, c_2 satisfy the equations

$$(a_{11} - r)c_1 + a_{12}c_2 = 0, \quad a_{21}c_1 + (a_{22} - r)c_2 = 0$$

r is called an *eigenvalue* of the matrix

They build solutions $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called an *eigenvector* associated with r

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

c = any number

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

c = any number

Eigenvalue $r = -0.8$

$$(a_{11} - r)c_1 + a_{12}c_2 = 0$$

$$a_{21}c_1 + (a_{22} - r)c_2 = 0$$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\ y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

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Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$0.8c_2 + 1.2c_2 = 0$$

Population Dynamic Application

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

~~$$0.8c_2 + 1.2c_2 = 0$$~~

Need solve only one equation.

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1.5c \\ c \end{pmatrix}$$

$c = \text{any real number}$

Population Dynamic Application

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$c_1 + 1.5c_2 = 0$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1.5c \\ c \end{pmatrix}$$

$c = \text{any real number}$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} -1.5 \times (-0.8)^n \\ (-0.8)^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and} \quad -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} -1.5 \times (-1)^n \times 0.8^n \\ (-1)^n \times 0.8^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$\begin{aligned}x_{n+1} &= 0.2x_n + 1.5y_n \\y_{n+1} &= 0.8x_n + 0.4y_n\end{aligned}\qquad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n c \begin{pmatrix} -1.5 \times 0.8^n \\ 0.8^n \end{pmatrix}$$

$c = \text{any number}$

Population Dynamic Application

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} c_1 r^n \\ c_2 r^n \end{pmatrix}$$

$$r^2 - 0.6r + 1.12 = 0 \quad \text{roots: } r = 1.4 \quad \text{and } -0.8$$

Eigenvalue $r = 1.4$

previous example

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = c \begin{pmatrix} 1.4^n \\ 0.8 \times 1.4^n \end{pmatrix}$$

$c = \text{any number}$

These solutions are unbounded

Eigenvalue $r = -0.8$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (-1)^n c \begin{pmatrix} -1.5 \times 0.8^n \\ 0.8^n \end{pmatrix}$$

$c = \text{any number}$

These tend to 0 !