

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, November 27

<http://math.arizona.edu/~cushing/250a.html>

Basics of Matrix Algebra

- What are matrices ?
- How does one work with them (i.e., what's their algebra) ?
- For what are they used ?

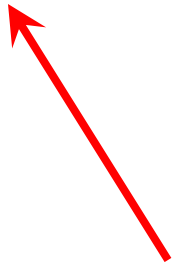
We'll motivate & develop matrix algebra in the context
of recursively defined sequences
(discrete dynamical systems).

$$X_{n+1} = rX_n$$



Population numbers
at start of n^{th} year

$$X_{n+1} = rX_n$$



Population numbers
at start of $(n+1)^{st}$ year

$$X_{n+1} = rX_n$$



Fraction surviving
one year

$$0 \leq r \leq 1$$

$$x_{n+1} = rx_n$$

Questions:

- Given an initial number $x_0 > 0$ calculate x_n for $n = 0, 1, 2, \dots$
- What is the long term fate of the population ?

$$\lim_{n \rightarrow +\infty} x_n = ?$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2x_0$$

$$x_3 = rx_2 = r^3x_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

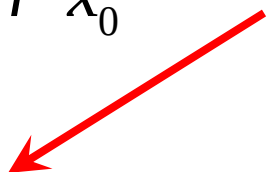
$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$\vdots$

$$x_n = r^n x_0$$

Observe pattern



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2x_0$$

$$x_3 = rx_2 = r^3x_0$$

$$\vdots$$

$$x_n = r^n x_0$$

$$\lim_{n \rightarrow +\infty} x_n = \begin{cases} 0 & \text{if } 0 \leq r < 1 \\ x_0 & \text{if } r = 1 \\ \infty & \text{if } r > 1 \end{cases}$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$\vdots$

$$x_n = r^n x_0$$

Often in applications
one wants to track
more than one quantity
(state variable).

$$x_{n+1} = rx_n$$

$$x_{n+1} =$$

$$x_1 = rx_0$$

$$y_{n+1} =$$

$$x_2 = rx_1 = r^2 x_0$$

Often in applications
one wants to track
more than one quantity
(state variable).

$$x_3 = rx_2 = r^3 x_0$$

$\vdots$

$$x_n = r^n x_0$$

Such problems lead to
two or more recursion formulas,
one for each state variable.

$$X_{n+1} = rX_n$$

$$X_{n+1} =$$

$$X_1 = rX_0$$

$$Y_{n+1} =$$

$$X_2 = rX_1 = r^2X_0$$

Often in applications
one wants to track
more than one quantity
(state variable).

$$X_3 = rX_2 = r^3X_0$$

$\vdots$

Such problems lead to
two or more recursion formulas,
one for each state variable.

$$X_n = r^n X_0$$

For example, suppose we follow
two classes of individuals through time :
say, *large (adults) & small (juveniles)*

$$x_{n+1} = rx_n$$

$$x_{n+1} =$$

$$x_1 = rx_0$$

$$y_{n+1} =$$

$$x_2 = rx_1 = r^2 x_0$$

x_n = small (juvenile), do not reproduce

$$x_3 = rx_2 = r^3 x_0$$

y_n = large (adults), reproduce, offspring are small

$\vdots$

$$x_n = r^n x_0$$

$$x_{n+1} = rx_n$$

$$x_{n+1} = (1 - g)x_n$$

$$x_1 = rx_0$$

$$y_{n+1} = gx_n$$

$$x_2 = rx_1 = r^2x_0$$

x_n = small (juvenile), do not reproduce

$$x_3 = rx_2 = r^3x_0$$

y_n = large (adults), reproduce, offspring are small

$\vdots$

All small individuals survive the year and
a fraction g grows into the larger class.

$$x_n = r^n x_0$$

$$x_{n+1} = rx_n$$

$$x_{n+1} = (1-g)x_n + by_n$$

$$x_1 = rx_0$$

$$y_{n+1} = gx_n$$

$$x_2 = rx_1 = r^2x_0$$

x_n = small (juvenile), do not reproduce

$$x_3 = rx_2 = r^3x_0$$

y_n = large (adults), reproduce, offspring are small

$\vdots$

All small individuals survive the year and
a fraction g grows into the larger class.

$$x_n = r^n x_0$$

Average number of small offspring b per adult.

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$\vdots$

$$x_n = r^n x_0$$

$$x_{n+1} = (1 - g)x_n + by_n$$

$$y_{n+1} = gx_n + fy_n$$

x_n = small (juvenile), do not reproduce

y_n = large (adults), reproduce, offspring are small

All small individuals survive the year and a fraction g grows into the larger class.

Average number of small offspring b per adult.

A fraction f of the larger class survives to reproduce the next year.

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$$\vdots$$

$$x_n = r^n x_0$$

$$x_{n+1} = (1-g)x_n + by_n$$

$$y_{n+1} = gx_n + f y_n$$

Example:

$$g = 0.8, \quad b = 1.5, \quad f = 0.4$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$\vdots$

$$x_n = r^n x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

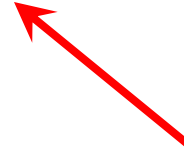
$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_{n+1} = (1-g)x_n + by_n$$

$$y_{n+1} = gx_n + fy_n$$

Example:

$$g = 0.8, \quad b = 1.5, \quad f = 0.4$$



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$$\vdots$$

$$x_n = r^n x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

Problems

- Find formulas for x_n, y_n in terms of $x_0, y_0 > 0$.
- Determine the long term fate of the population :

$$\lim_{n \rightarrow +\infty} x_n = ? \quad \lim_{n \rightarrow +\infty} y_n = ?$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

⋮

$$x_n = r^n x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

Problems

- Find formulas for x_n, y_n in terms of $x_0, y_0 > 0$.
- Determine the long term fate of the population :

$$\lim_{n \rightarrow +\infty} x_n = ?$$

$$\lim_{n \rightarrow +\infty} y_n = ?$$

Try same procedure:

Iterate and look for a pattern.

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2x_1 + 1.5y_1$$

$$y_2 = 0.8x_1 + 0.4y_1$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

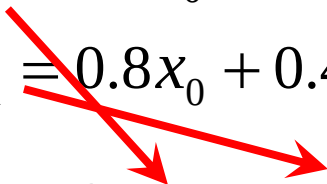
$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2x_1 + 1.5y_1$$

$$y_2 = 0.8x_1 + 0.4y_1$$



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

$$y_2 = 0.8[0.2x_0 + 1.5y_0] + 0.4[0.8x_0 + 0.4y_0]$$

Next: gather x_0 and y_0 terms together

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 2 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 2 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 1.24x_0 + 0.9y_0$$

$$y_2 = 0.48x_0 + 1.36y_0$$

Repeat procedure for 3rd year.

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 1.24x_0 + 0.9y_0$$

$$y_2 = 0.48x_0 + 1.36y_0$$

$$x_3 = 0.968x_0 + 2.22y_0$$

$$y_3 = 1.184x_0 + 1.264y_0$$

... it turns out

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 1.24x_0 + 0.9y_0$$

$$y_2 = 0.48x_0 + 1.36y_0$$

$$x_3 = 0.968x_0 + 2.22y_0$$

$$y_3 = 1.184x_0 + 1.264y_0$$

$$x_n = ?$$

$$y_n = ?$$

See a pattern ?

(Probably not !)

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 1.24x_0 + 0.9y_0$$

$$y_2 = 0.48x_0 + 1.36y_0$$

$$x_3 = 0.968x_0 + 2.22y_0$$

$$y_3 = 1.184x_0 + 1.264y_0$$

$$x_n = [?]x_0 + [?]y_0$$

$$y_n = [?]x_0 + [?]y_0$$

At least it seems
we can say this:

in year n the terms
have this form.

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

$$y_2 = 0.8[0.2x_0 + 1.5y_0] + 0.4[0.8x_0 + 0.4y_0]$$

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$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

$$y_2 = 0.8[0.2x_0 + 1.5y_0] + 0.4[0.8x_0 + 0.4y_0]$$

gather x_0 and y_0 terms together

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

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$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

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gather x_0 and y_0 terms together

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

$$y_2 = 0.8[0.2x_0 + 1.5y_0] + 0.4[0.8x_0 + 0.4y_0]$$

gather x_0 and y_0 terms together

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = 0.2[0.2x_0 + 1.5y_0] + 1.5[0.8x_0 + 0.4y_0]$$

$$y_2 = 0.8[0.2x_0 + 1.5y_0] + 0.4[0.8x_0 + 0.4y_0]$$

gather x_0 and y_0 terms together

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

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$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

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$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

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$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

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$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}$$

place the coefficients
in an array called a

2×2 matrix

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

plays the
role of r

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$R = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$R = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}$$

these coefficients also
make a 2×2 *matrix*

$$\begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

plays the
role of r^2

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$R = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_1 = 0.2x_0 + 1.5y_0$$

$$y_1 = 0.8x_0 + 0.4y_0$$

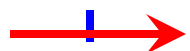
$$x_2 = [0.2 \times 0.2 + 1.5 \times 0.8]x_0 + [0.2 \times 1.5 + 1.5 \times 0.4]y_0$$

$$y_2 = [0.8 \times 0.2 + 0.4 \times 0.8]x_0 + [0.8 \times 1.5 + 0.4 \times 0.4]y_0$$

This suggests we organize our calculations as follows:

$$R = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}$$

Should be
 R^2 !



$$R^2 = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Subscript notation

$$a_{ij}$$

First subscript i denotes row.

Second subscript j denotes column.

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \\ & \end{pmatrix}$$

first row, *first column* position uses

first row of a 's, *first column* of b 's

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

first row, *second column* position uses

first row of a 's, *second column* of b 's

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

second row, *first column* position uses

second row of a 's, *first column* of b 's

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

second row, *second column* position uses

second row of a 's, *second column* of b 's

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Procedure works for matrices of different sizes ... as long as

columns of a 's matrix = # rows of b 's matrix

This suggests that matrix multiplication should be defined so that

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.2 \times 0.2 + 1.5 \times 0.8 & 0.2 \times 1.5 + 1.5 \times 0.4 \\ 0.8 \times 0.2 + 0.4 \times 0.8 & 0.8 \times 1.5 + 0.4 \times 0.4 \end{pmatrix}$$

Definition of Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -0.3 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & \\ -0.3 & \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & 0.2 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 2.2 & -3 \\ -1.5 & 3 \end{pmatrix}$$

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & 0.2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 2.2 & -3 \\ -1.5 & 3 \end{pmatrix}$$

not the same !

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -11 \\ -0.3 & 0.2 \end{pmatrix}$$


$$\begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} = \begin{pmatrix} 2.2 & -3 \\ -1.5 & 3 \end{pmatrix}$$

not the same !



In general, matrix multiplication *does not commute*.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \text{does not} \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

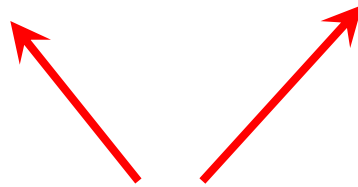
necessarily equal

Examples

$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix}$$

Examples

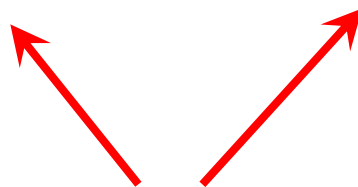
$$\begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 0.1 & 0 \end{pmatrix}$$



Same !

Examples

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

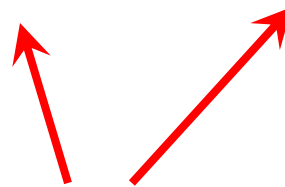


Same !

Examples

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$



Same !

Examples

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is called the 2×2 *identity matrix*

Plays the role that 1 does for numbers.

$$x_{n+1} = rx_n$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

$$x_{n+1} = rx_n$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

Note:



$$\begin{pmatrix} 0.2x_n + 1.5y_n \\ 0.8x_n + 0.4y_n \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$x_{n+1} = rx_n$$

$$x_{n+1} = 0.2x_n + 1.5y_n$$

$$y_{n+1} = 0.8x_n + 0.4y_n$$

We can write the equations in matrix form

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Then we can iterate using matrix multiplication
as follows ...

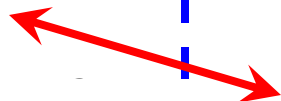
$$x_{n+1} = rx_n \quad \longleftrightarrow \quad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

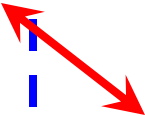
$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

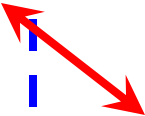
$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

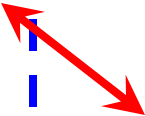
$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

$$x_1 = rx_0$$

$$x_2 = rx_1 = r^2 x_0$$

$$x_3 = rx_2 = r^3 x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^3 \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$



$$x_{n+1} = rx_n$$

$$x_n = r^n x_0$$

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Questions:

How do we efficiently calculate powers of matrices ?

How do we use this solution formula to calculate

$$\lim_{n \rightarrow +\infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \lim_{n \rightarrow +\infty} x_n \\ \lim_{n \rightarrow +\infty} y_n \end{pmatrix} \quad ??$$

$$x_{n+1} = rx_n$$

$$x_n = r^n x_0$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Note: *not* the same

$$\begin{pmatrix} 0.2^n & 1.5^n \\ 0.8^n & 0.4^n \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Examples:

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 = \begin{pmatrix} 1.24 & 0.90 \\ 0.48 & 1.36 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^3 = \begin{pmatrix} 0.97 & 2.22 \\ 1.18 & 1.26 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^{10} = \begin{pmatrix} 13.27 & 19.65 \\ 10.48 & 15.83 \end{pmatrix}$$

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^{100} = \begin{pmatrix} 1.86 \times 10^{14} & 2.80 \times 10^{14} \\ 1.49 \times 10^{14} & 2.24 \times 10^{14} \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Start with 5 (thousand) juveniles and
4 (thousand) adults

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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Examples:

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1.24 & 0.90 \\ 0.48 & 1.36 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 9.80 \\ 7.84 \end{pmatrix}$$

2 years

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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2 years

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^3 \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 0.97 & 2.22 \\ 1.18 & 1.26 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 13.72 \\ 10.98 \end{pmatrix}$$

3 years

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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3 years

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 13.27 & 19.65 \\ 10.48 & 15.83 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 144.6 \\ 115.7 \end{pmatrix}$$

10 years

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^2 \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1.24 & 0.90 \\ 0.48 & 1.36 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 9.80 \\ 7.84 \end{pmatrix}$$

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10 years

$$\begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix}^{100} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 1.86 \times 10^{14} & 2.80 \times 10^{14} \\ 1.49 \times 10^{14} & 2.24 \times 10^{14} \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 2.05 \times 10^{15} \\ 1.64 \times 10^{15} \end{pmatrix}$$

100 years

Appears to grow without bound.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.2 & 1.5 \\ 0.8 & 0.4 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

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There's another method
for calculating the solution,

but we need to know some
more about matrices ...



Matrices and Linear Algebraic Equations

We can write the problem of two linear equations in two unknowns

$$a_{11}x + b_{12}y = b_1$$

$$a_{21}x + b_{22}y = b_2$$

in matrix notation as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$a x = b$$

The solution of
one equation
in one unknown

$$x = \frac{b}{a}$$



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

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The solution of
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Suggests we should
consider matrix division

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

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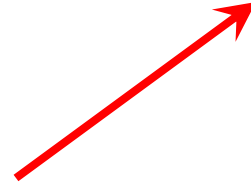
$$x = \frac{b}{a}$$



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$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}$$

But what sense can
be made out of this ?



$$a x = b$$

The solution of
one equation
in one unknown

$$x = \frac{b}{a}$$



$$x = a^{-1}b$$

Division defined in
terms of multiplication

Suggests we should
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By analogy $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$a x = b$$

The solution of
one equation
in one unknown

$$x = \frac{b}{a}$$



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Division defined in
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But what sense can
be made out of this ?

$$a x = b$$

The solution of
one equation
in one unknown

$$x = \frac{b}{a}$$

$$x = a^{-1}b$$

How is a^{-1} defined ?

Suggests we should
consider matrix division

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The solution of
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$$x = a^{-1}b$$

How is a^{-1} defined ?

a^{-1} is the one and only number
such that $a^{-1}a = 1$

Suggests we should
consider matrix division

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

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By analogy

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But what sense can
be made out of this ?

a^{-1} is the one and only number
such that $a^{-1}a = 1$

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1}$ is a matrix such that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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called the *multiplicative inverse*

The solution of

$$a x = b$$

is

$$x = a^{-1}b$$

The solution of

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Questions

Does every matrix have an inverse ?

If a matrix has an inverse, how do we calculate it ?

If not, does this mean the problem have no solution ?