

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, November 15

<http://math.arizona.edu/~cushing/250a.html>

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

The Binomial Series

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n \quad \text{for } -1 < x < 1$$

p is any constant (not necessarily an integer)

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots \quad \text{for } -1 < x < 1$$

Example

$$\sqrt{1+x} = (1+x)^{1/2}$$

$$p = 1/2$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots \quad \text{for } -1 < x < 1$$

Example

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6}x^3 + \dots$$

$$p = 1/2$$

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$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots \quad \text{for } -1 < x < 1$$

10.3 Other Ways of Calculating Taylor Polynomials

Substitution

(composite functions)

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Recall: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for } |x| < 1$

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Recall: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ for $|x| < 1$

$$\frac{1}{1+x^2} = ?$$

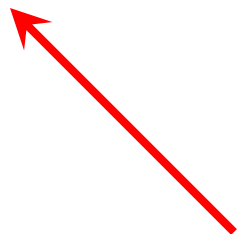
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Substitute $-x^2$ for x .

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$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots + (-x^2)^n \cdots \quad \text{for } |-x^2| < 1$$

Substitute $-x^2$ for x .

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$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \quad \text{for } |x| < 1$$

10.3 Other Ways of Calculating Taylor Polynomials

Substitution

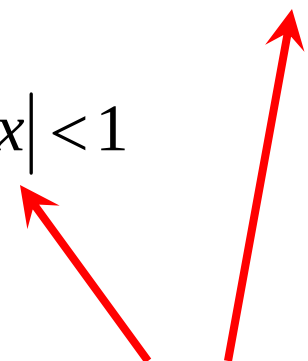
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equivalent



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Substitution

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Recall: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$ for $|x| < 1$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \text{ for all } x$$

$$\frac{1}{1 - \sin x} = ?$$

Find the 3rd degree Taylor polynomial

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$$\frac{1}{1-\sin x} = 1 + \sin x + (\sin x)^2 + (\sin x)^3 + \cdots \quad \text{for } |\sin x| < 1$$

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3rd degree or less

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Only term 3rd degree or less is x^2

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$$+ \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \right)^3 + \cdots$$

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$$\frac{1}{1-\sin x} \approx 1 + x + x^2 + \frac{5}{6}x^3 = p_3(x)$$

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Power series can be integrated and differentiated term-wise
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Thus:
$$\int \frac{1}{1+x^2} dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + \frac{(-1)^n}{2n+1}x^{2n+1} + \cdots$$

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$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| < 1$$

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Thus:
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$$-\frac{2x}{(1+x^2)^2} = \sum_{n=1}^{\infty} 2n(-1)^n x^{2n-1} \quad \text{for } |x| < 1$$

$$\frac{x}{(1+x^2)^2} = \sum_{n=1}^{\infty} n(-1)^{n+1} x^{2n-1} \quad \text{for } |x| < 1$$

Question: How good an approximation is $p_n(x)$?

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

For example, how accurate are these approximations ?

$$\ln x \approx x - 1 = p_1(x)$$

$$\sin x \approx x = p_1(x)$$

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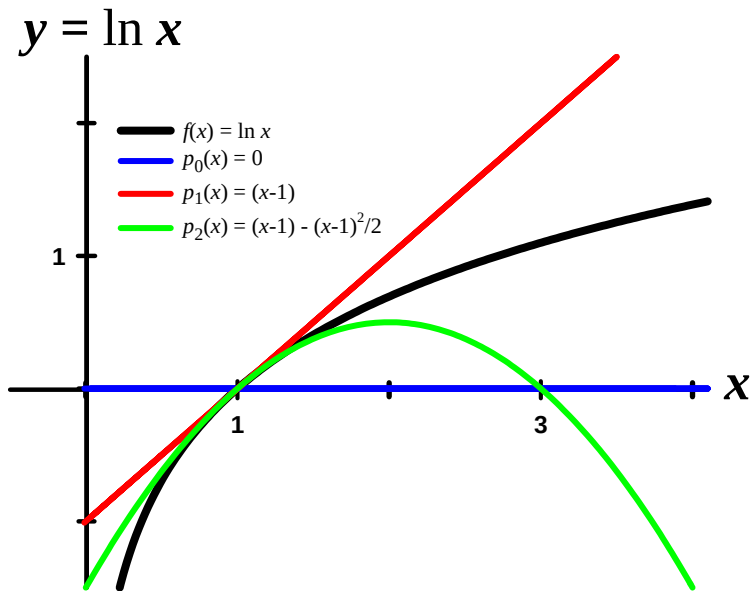
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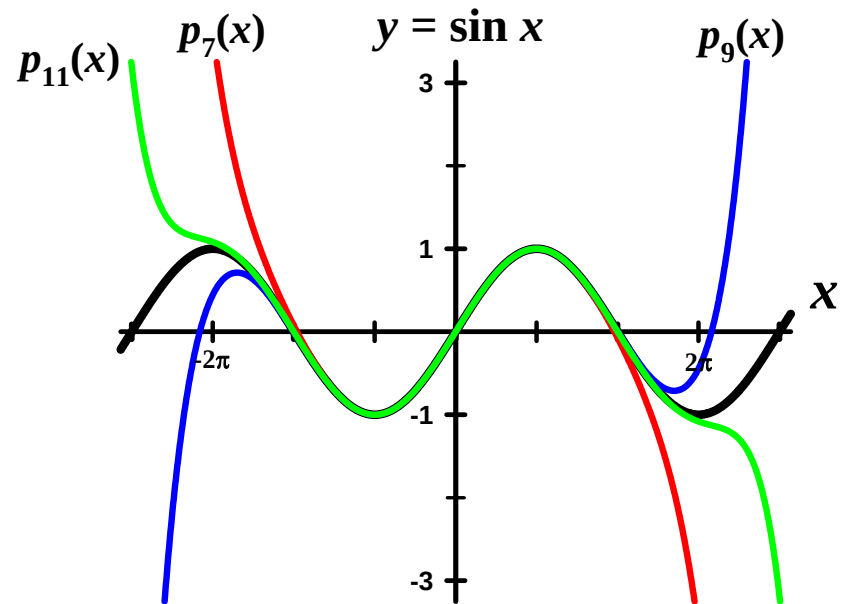
$$\ln x \approx x-1 = p_1(x)$$

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$$\sin x \approx x - \frac{1}{3!}x^3 = p_3(x)$$



Question: How good an approximation is $p_n(x)$?

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Clearly the answer depends on both n and x .

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Lagrange's Error Bound (Theorem 10.1, text)

$$|f(x) - p_n(x)| \leq$$

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Lagrange's Error Bound (Theorem 10.1, text)

$$|f(x) - p_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

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$$|f(x) - p_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

where M is a number satisfying

$$|f^{(n+1)}| \leq M \quad \text{on the interval between } a \text{ and } x$$

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

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Example

$$\sin x \approx x = p_1(x)$$

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$$\sin x \approx x = p_1(x)$$

$$|\sin x - x| \leq \frac{M}{2} x^2$$

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Example

$$\sin x \approx x = p_1(x) \qquad |\sin x - x| \leq \frac{M}{2} x^2$$

$$f^{(2)} = -\sin x \quad \text{and so} \quad |f^{(2)}| \leq 1$$

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

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Example

$$\sin x \approx x = p_1(x) \qquad |\sin x - x| \leq \frac{1}{2} x^2$$

For example, take $x = 0.1$

$$|\sin 0.1 - 0.1| \leq \frac{1}{2} (0.1)^2$$

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

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For example, take $x = 0.1$

$$|\sin 0.1 - 0.1| \leq \frac{1}{2} (0.1)^2 = 0.005$$

$$\sin 0.1 = 0.1 \pm 0.005$$

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

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Example

$$\sin x \approx x - \frac{1}{3!} x^3 = p_3(x) \quad \left| \sin x - \left(x - \frac{1}{3!} x^3 \right) \right| \leq \frac{1}{4!} x^4$$

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$$\sin x \approx x - \frac{1}{3!} x^3 = p_3(x) \quad \left| \sin x - \left(x - \frac{1}{3!} x^3 \right) \right| \leq \frac{1}{4!} x^4$$

For example, take $x = 0.1$

$$\left| \sin 0.1 - \left(0.1 - \frac{1}{6} (0.1)^3 \right) \right| \leq \frac{1}{24} (0.1)^4$$

$$f(x) \approx f(a) + f^{(1)}(a)(x-a) + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

$$|f(x) - p_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$$

where M is a number satisfying $|f^{(n+1)}| \leq M$ on the interval between a and x

Example

$$\sin x \approx x - \frac{1}{3!} x^3 = p_3(x) \quad \left| \sin x - \left(x - \frac{1}{3!} x^3 \right) \right| \leq \frac{1}{4!} x^4$$

For example, take $x = 0.1$

$$|\sin 0.1 - 0.09983| \leq 4.167 \times 10^{-6}$$

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$$\sin 0.1 = 0.099833 \pm 4.167 \times 10^{-6}$$

Application



A thin disk of radius a
and mass M

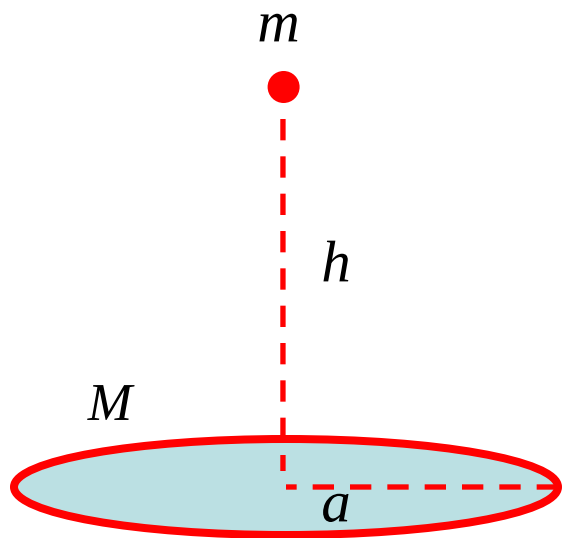
Application



A particle of mass m
height h above center



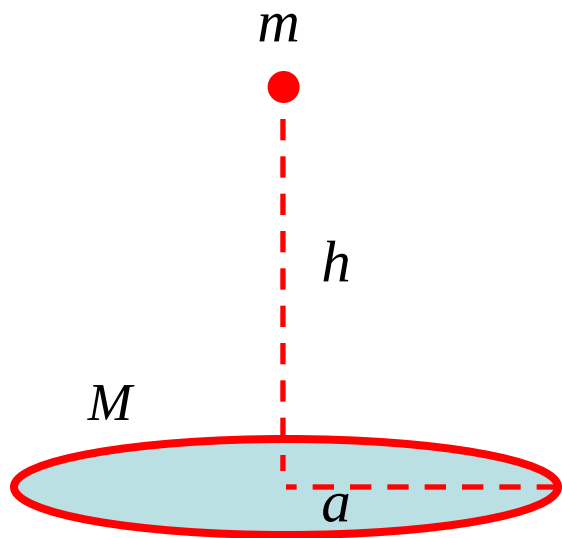
A thin disk of radius a
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Application

$$F = \frac{2gMmh}{a^2} \left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}} \right)$$

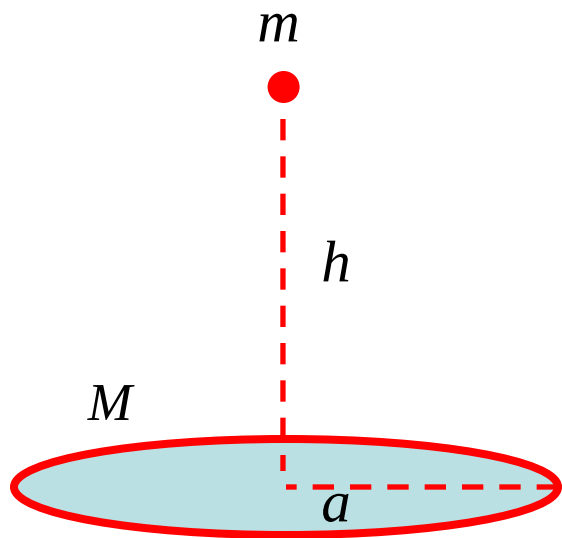
Gravitational force exerted by the disk
on the particle .



Application

$$F = \frac{2gMmh}{a^2} \left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}} \right)$$

Problem: approximate F for large h
(in particular, for $h > a$)

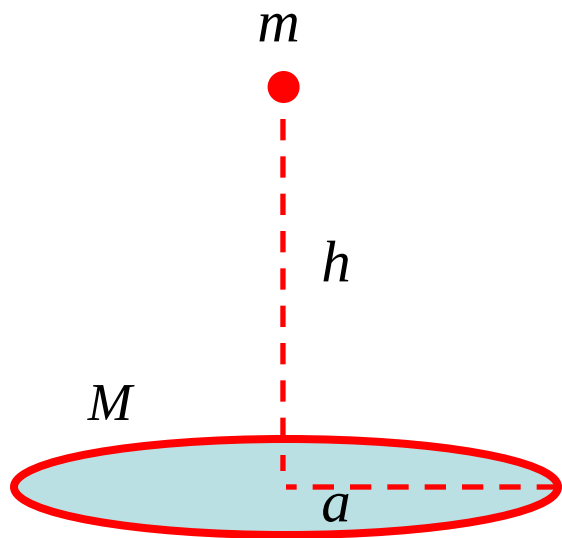


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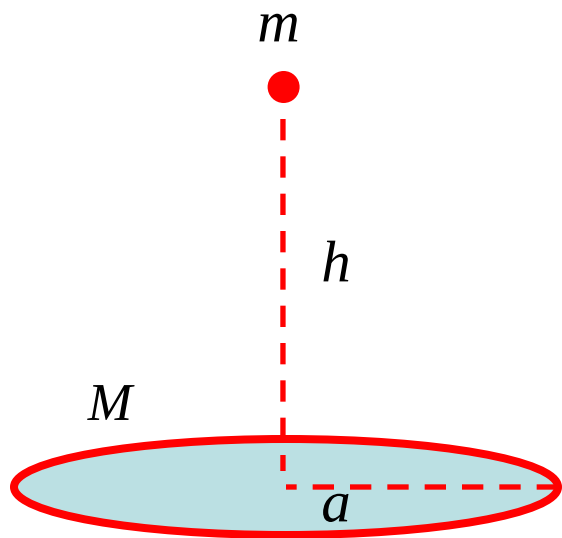
Method: consider F as a function of
 $x = a / h$
and calculate low order Taylor polynomials in x
centered at 0.



Application

$$F = \frac{2gMmh}{a^2} \left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}} \right)$$

$$x = \frac{a}{h} \Rightarrow h = \frac{a}{x}$$

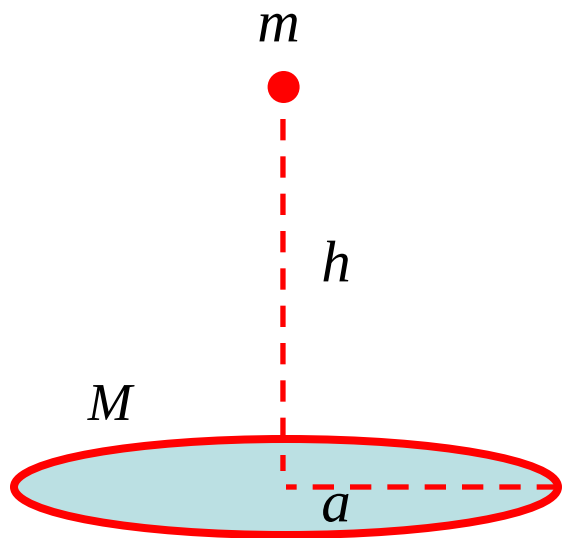


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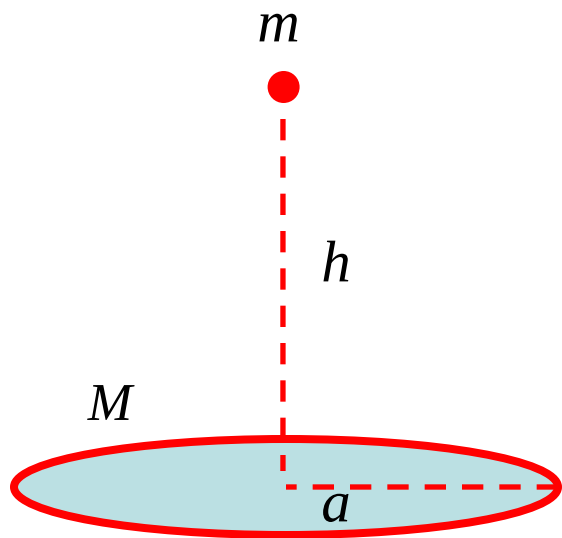


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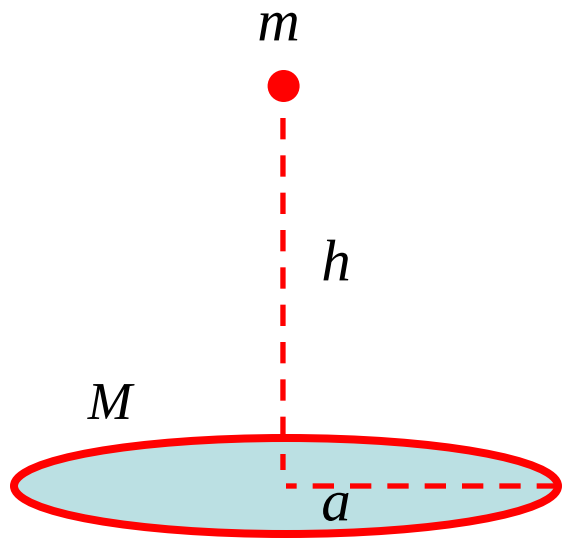


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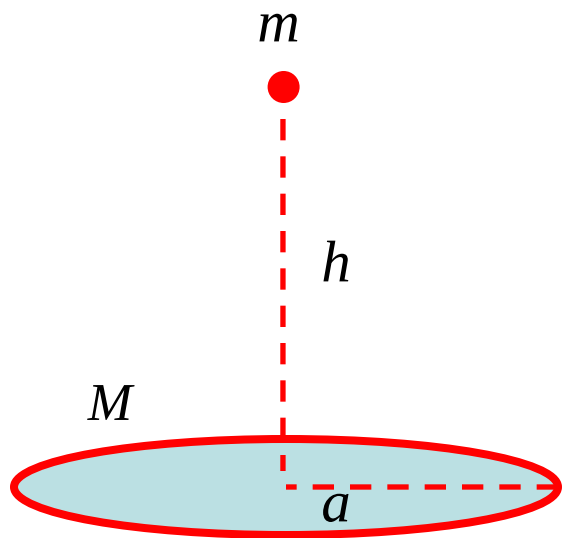


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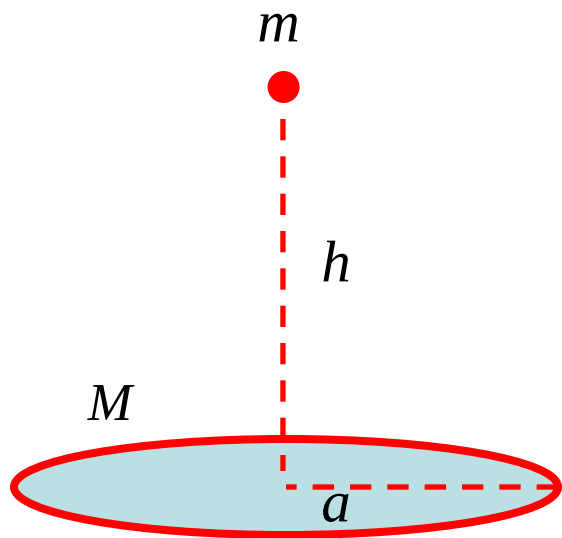
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$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots \quad \text{for } -1 < x < 1$$



Application

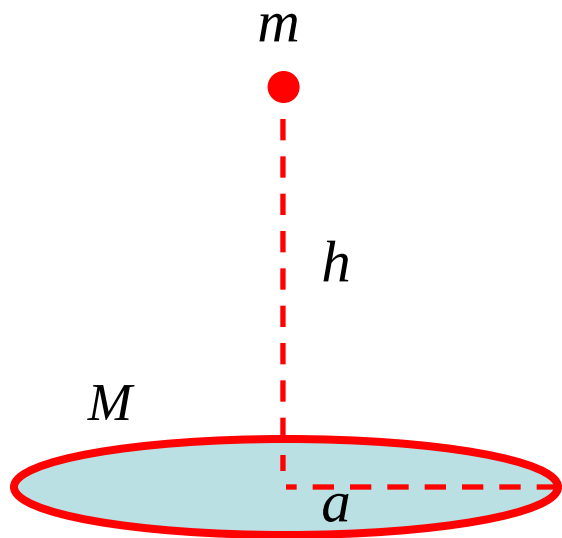
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substitution of x^2



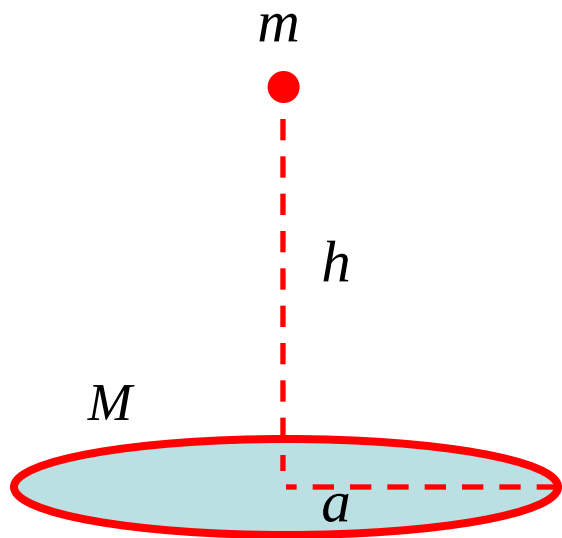
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$$(1 + x^2)^{-1/2} = 1 - \frac{1}{2}x^2 + \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)}{2!}x^4 + \dots \quad \text{for } -1 < x < 1$$



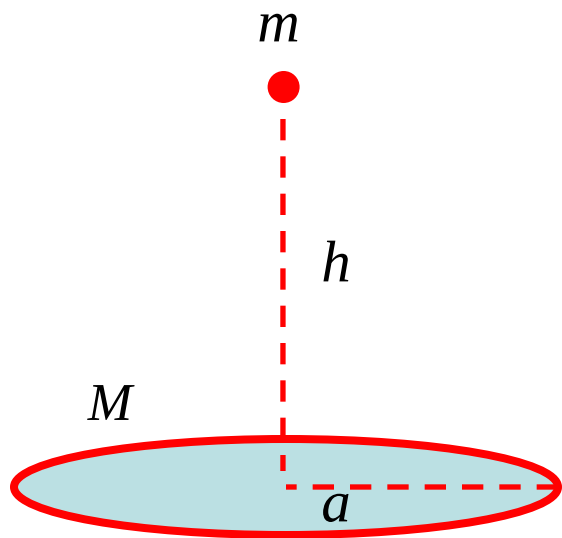
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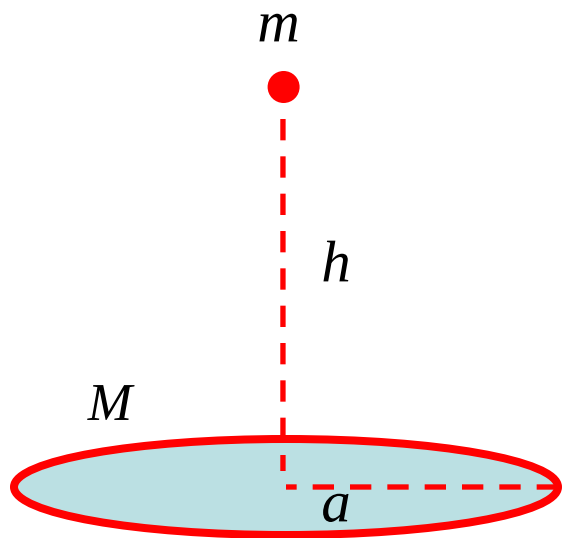
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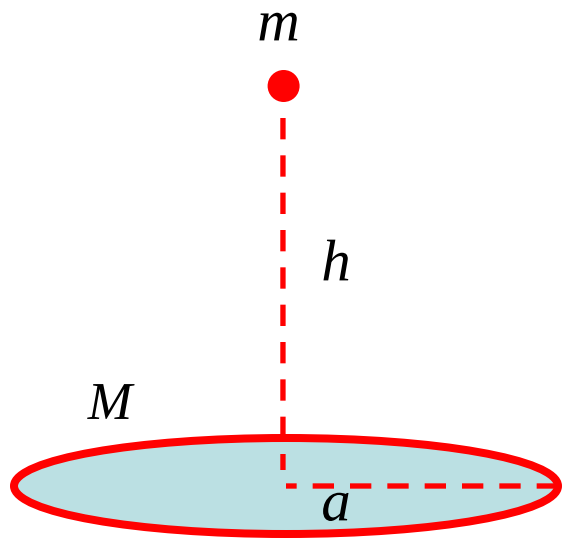
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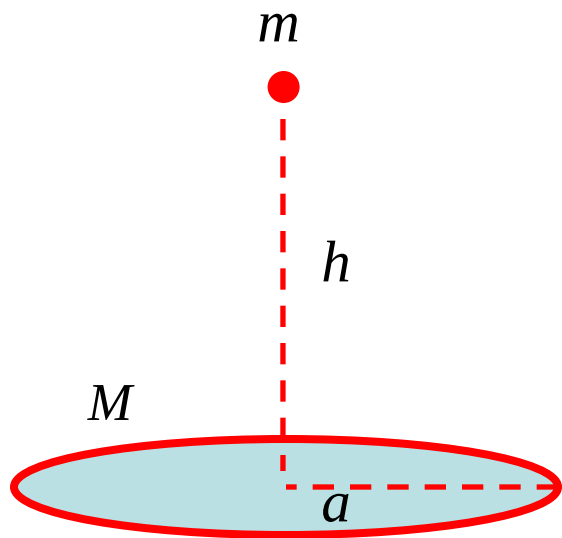
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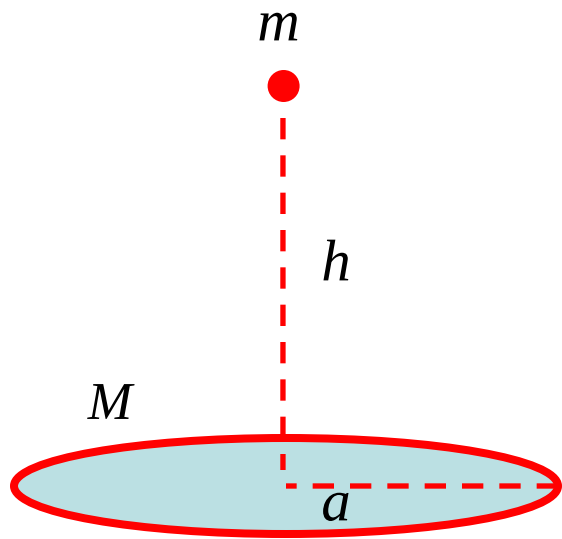
$$F = \frac{2gMm}{a^2} \left(\frac{1}{2} \frac{a^2}{h^2} - \frac{3}{8} \frac{a^4}{h^4} + \dots \right)$$



Application

$$F = \frac{2gMmh}{a^2} \left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}} \right)$$

$$F = \frac{gMm}{h^2} \left(1 - \frac{3}{4} \frac{a^2}{h^2} + \dots \right) \quad \text{for} \quad 0 < \frac{a}{h} < 1$$



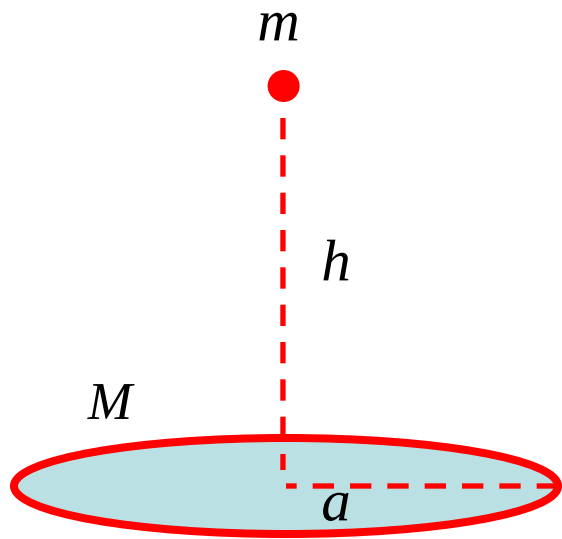
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To lowest order :

$$F \approx \frac{gMm}{h^2} \quad \text{for } h \text{ large}$$



Application

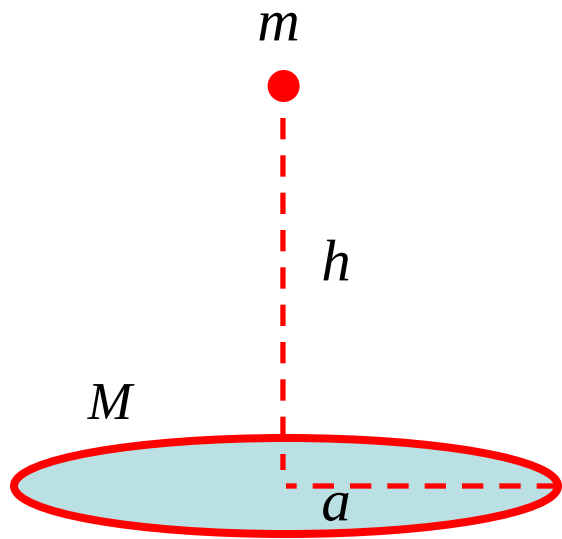
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To lowest order :

Does not
depend on a !

$$F \approx \frac{gMm}{h^2} \quad \text{for } h \text{ large}$$



Application

$$F = \frac{2gMmh}{a^2} \left(\frac{1}{h} - \frac{1}{(a^2 + h^2)^{1/2}} \right)$$

$$F = \frac{gMm}{h^2} \left(1 - \frac{3}{4} \frac{a^2}{h^2} + \dots \right) \quad \text{for } 0 < \frac{a}{h} < 1$$

To lowest order :

$$F \approx \frac{gMm}{h^2} \quad \text{for } h \text{ large}$$

To next order :

higher order approximation depends on a

$$F \approx \frac{gMm}{h^2} \left(1 - \frac{3}{4} \frac{a^2}{h^2} \right) \quad \text{for } h \text{ large}$$