

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

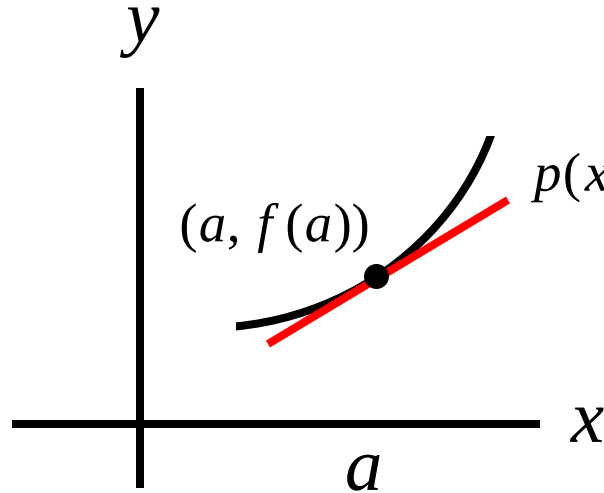
Tuesday, November 13

<http://math.arizona.edu/~cushing/250a.html>

First Degree Polynomial Approximation

Approximate $y = f(x)$ at $x = a$ by a first degree polynomial.

Geometrically, approximate graph of $f(x)$ by that of its tangent at $x = a$.



tangent line:

$$p(x) = f(a) + f'(a)(x - a)$$

$p(x)$ passes through same point as $f(x)$.
and has same slope as $f(x)$.

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

First degree Taylor polynomial of $f(x)$ at $x = a$.

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$p_1(x) = 1 + x$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x-a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$p_1(x) = 1 + x \approx e^x \text{ for } x \approx 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$f(0) = 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$f(0) = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = 1$$

$$p_1(x) = x \approx \sin x \text{ for } x \approx 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \ln x \text{ at } a = 1$$

$$f(1) = 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \ln x \text{ at } a = 1$$

$$f(1) = 0$$

$$f'(x) = 1/x \Rightarrow f'(1) = 1$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = \ln x \text{ at } a = 1$$

$$f(1) = 0$$

$$f'(x) = 1/x \Rightarrow f'(1) = 1$$

$$p_1(x) = x - 1 \approx \ln x \text{ for } x \approx 1$$

If the graph of $f(x)$ is significantly curved at $x = a$,
the tangent will be a poor approximation.

A better approximation would try to approximate some of f 's curvature.

If the graph of $f(x)$ is significantly curved at $x = a$,
the tangent will be a poor approximation.

A better approximation would try to approximate some of f 's curvature.

Graphs of *second* degree polynomials are parabolas.

IDEA: match f , f' and the *second* derivative f''
with a quadratic polynomial.

Second Degree Polynomial Approximation

Goal: find a quadratic polynomial $p(x)$ such that

$$p(a), \quad p'(a), \quad p''(a)$$

equal respectively

$$f(a), \quad f'(a), \quad f''(a)$$

Second Degree Polynomial Approximation

Goal: find a quadratic polynomial $p(x)$ such that

$$p(a), \quad p'(a), \quad p''(a)$$

equal respectively

$$f(a), \quad f'(a), \quad f''(a)$$

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

Second Degree Polynomial Approximation

Goal: find a quadratic polynomial $p(x)$ such that

$$p(a), \quad p'(a), \quad p''(a)$$

equal respectively

$$f(a), \quad f'(a), \quad f''(a)$$

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

$$p'(x) = c_1 + 2c_2(x - a)$$

$$p''(x) = 2c_2$$

Second Degree Polynomial Approximation

Goal: find a quadratic polynomial $p(x)$ such that

$$p(a), \quad p'(a), \quad p''(a)$$

equal respectively

$$f(a), \quad f'(a), \quad f''(a)$$

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

$$p(a) = c_0$$

$$p'(a) = c_1$$

$$p''(a) = 2c_2$$

Second Degree Polynomial Approximation

Goal: find a quadratic polynomial $p(x)$ such that

$$p(a), \quad p'(a), \quad p''(a)$$

equal respectively

$$f(a), \quad f'(a), \quad f''(a)$$

$$p(x) = c_0 + c_1(x - a) + c_2(x - a)^2$$

$$p(a) = c_0 \qquad c_0 = f(a)$$

$$p'(a) = c_1 \quad \Rightarrow \quad c_1 = f'(a)$$

$$p''(a) = 2c_2 \qquad c_2 = f''(a) / 2$$

Second Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a)$$

$$p_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2$$

Second degree Taylor polynomial of $f(x)$ at $x = a$.

Second Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a)$$

$$p_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)(x - a)^2 \approx f(x) \text{ for } x \approx a$$

Second degree Taylor polynomial of $f(x)$ at $x = a$.

Second Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x-a)$$

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 \approx f(x) \text{ for } x \approx a$$

Second degree Taylor polynomial of $f(x)$ at $x = a$.

Note: first degree approximation is contained
in second degree approximations

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$\begin{array}{l} f'(x) = e^x \\ f''(x) = e^x \end{array} \Rightarrow \begin{array}{l} f'(0) = 1 \\ f''(0) = 1 \end{array} \Rightarrow p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$\begin{array}{l} f'(x) = e^x \\ f''(x) = e^x \end{array} \Rightarrow \begin{array}{l} f'(0) = 1 \\ f''(0) = 1 \end{array} \Rightarrow p_2(x) = 1 + x + \frac{1}{2} x^2$$

We expect this to be a
better approximation
to e^x than $p_1(x)$

Second Degree Polynomial Approximation

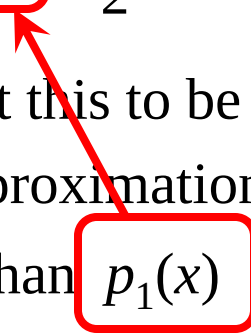
$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$\begin{array}{l} f'(x) = e^x \\ f''(x) = e^x \end{array} \Rightarrow \begin{array}{l} f'(0) = 1 \\ f''(0) = 1 \end{array} \Rightarrow p_2(x) = 1 + x + \frac{1}{2} x^2$$

We expect this to be a better approximation to e^x than $p_1(x)$



Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

Sequence of Taylor Polynomials

$$p_0(x) = 1$$

$$p_1(x) = 1 + x$$

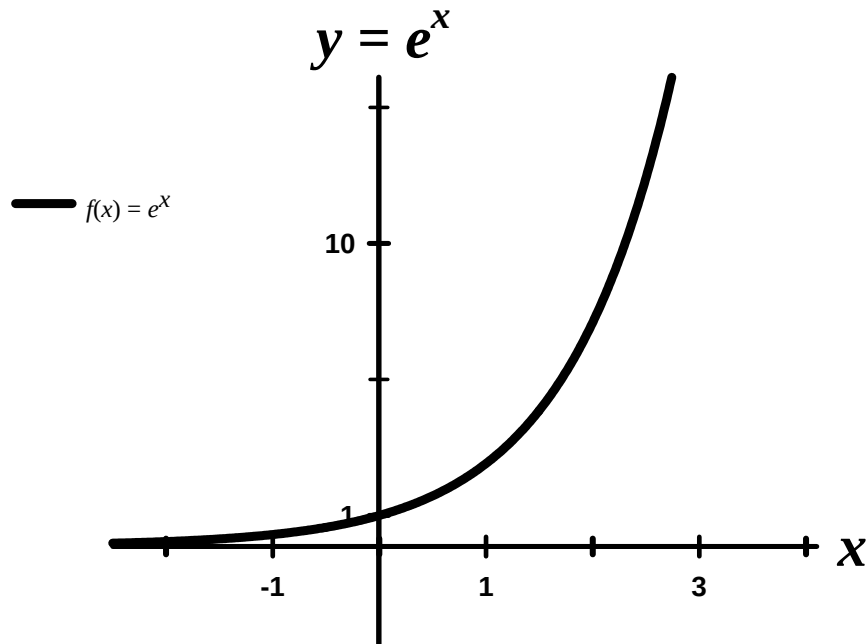
$$p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \text{ at } a = 0$$



Sequence of Taylor Polynomials

$$p_0(x) = 1$$

$$p_1(x) = 1 + x$$

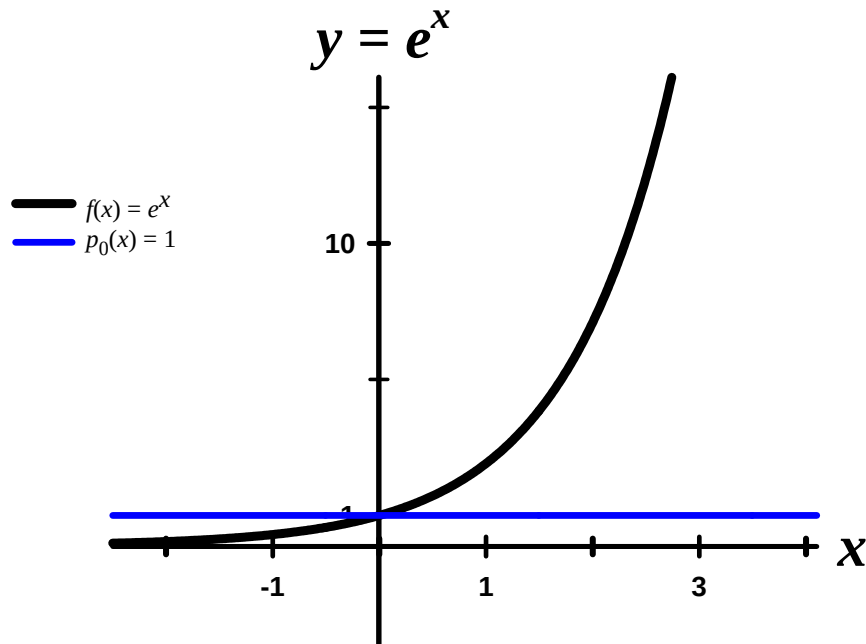
$$p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \text{ at } a = 0$$



Sequence of Taylor Polynomials

$$p_0(x) = 1$$

$$p_1(x) = 1 + x$$

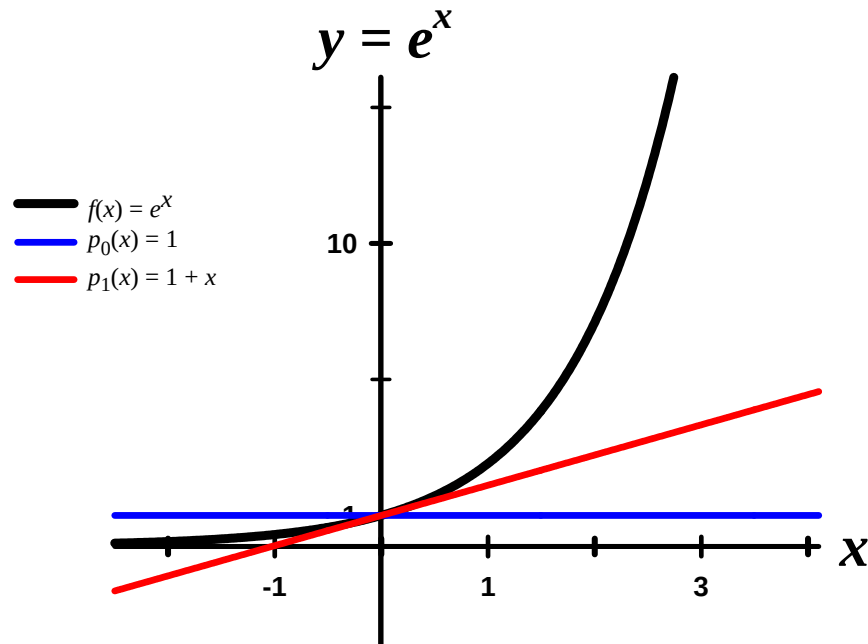
$$p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \text{ at } a = 0$$



Sequence of Taylor Polynomials

$$p_0(x) = 1$$

$$p_1(x) = 1 + x$$

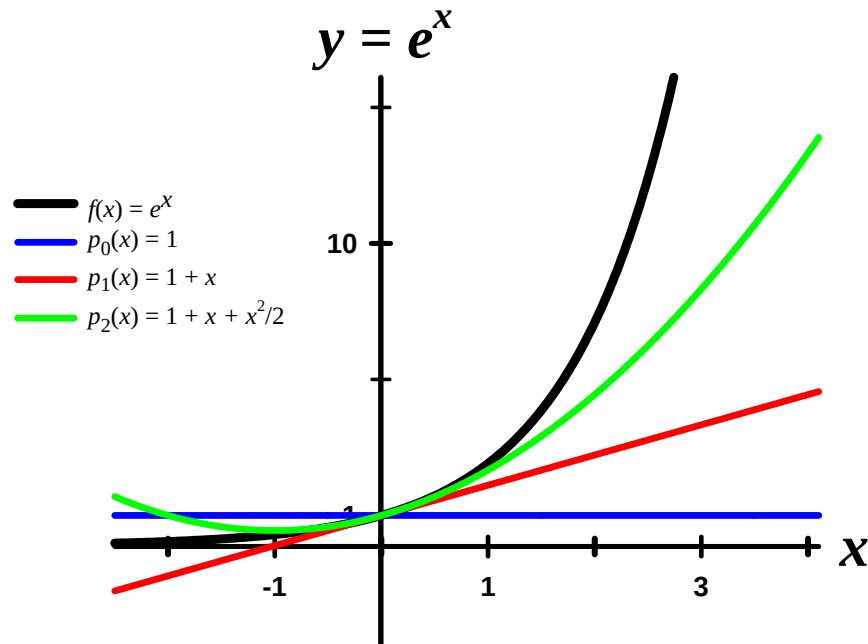
$$p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = e^x \text{ at } a = 0$$



Sequence of Taylor Polynomials

$$p_0(x) = 1$$

$$p_1(x) = 1 + x$$

$$p_2(x) = 1 + x + \frac{1}{2} x^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$\begin{array}{ll} f'(x) = \cos x & \\ f''(x) = -\sin x & \end{array} \Rightarrow \begin{array}{l} f'(0) = 1 \\ f''(0) = 0 \end{array}$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$\begin{array}{lll} f'(x) = \cos x & \Rightarrow & f'(0) = 1 \\ f''(x) = -\sin x & \Rightarrow & f''(0) = 0 \end{array} \quad \Rightarrow \quad p_2(x) = x$$

In this case, $p_2(x)$ is the same as $p_1(x)$!

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$\begin{array}{lcl} f'(x) = 1/x & & f'(1) = 1 \\ f''(x) = -1/x^2 & \Rightarrow & f''(1) = -1 \end{array}$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

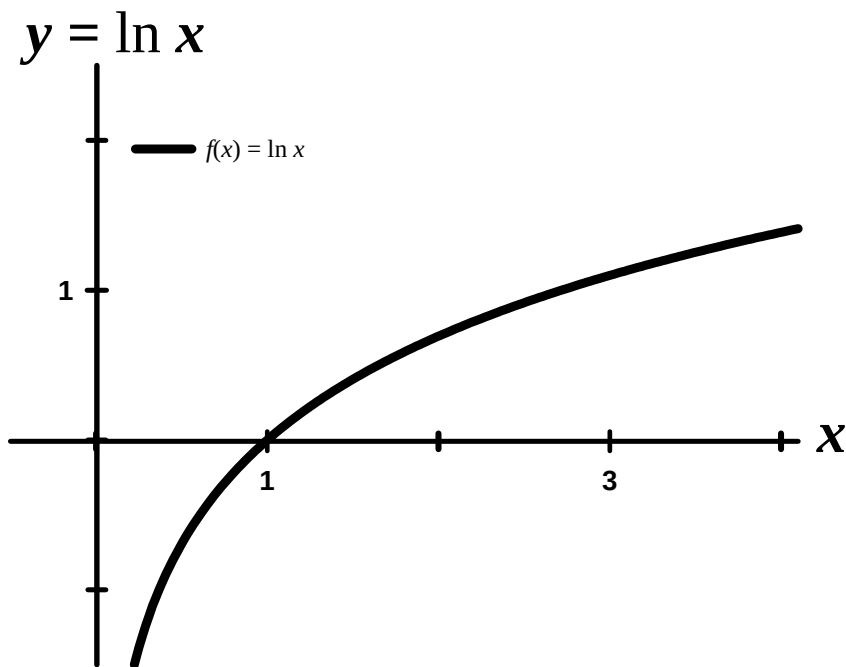
$$\begin{array}{l} f'(x) = 1/x \\ f''(x) = -1/x^2 \end{array} \Rightarrow \begin{array}{l} f'(1) = 1 \\ f''(1) = -1 \end{array} \Rightarrow p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$



Sequence of Taylor Polynomials

$$p_0(x) = 0$$

$$p_1(x) = (x-1)$$

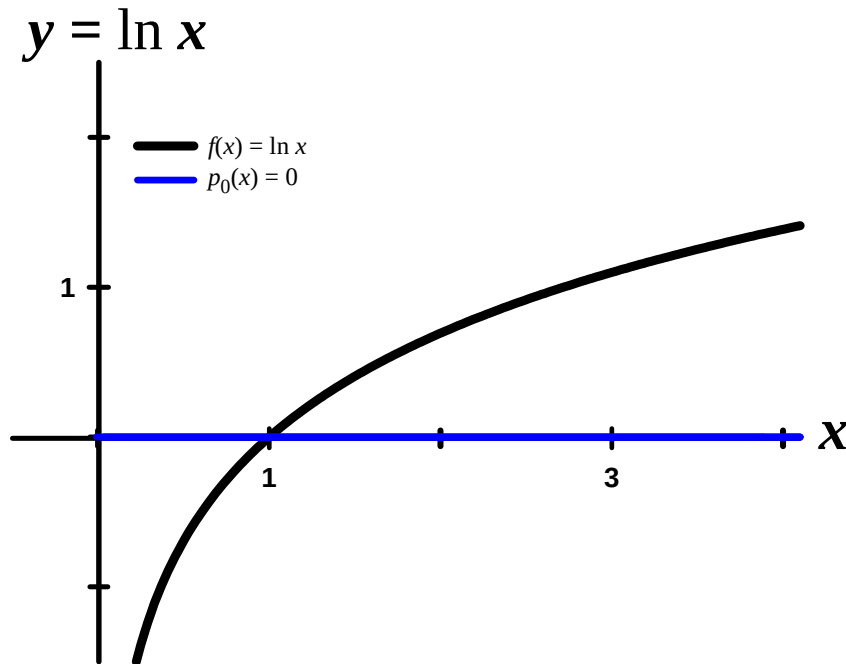
$$p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$



Sequence of Taylor Polynomials

$$p_0(x) = 0$$

$$p_1(x) = (x-1)$$

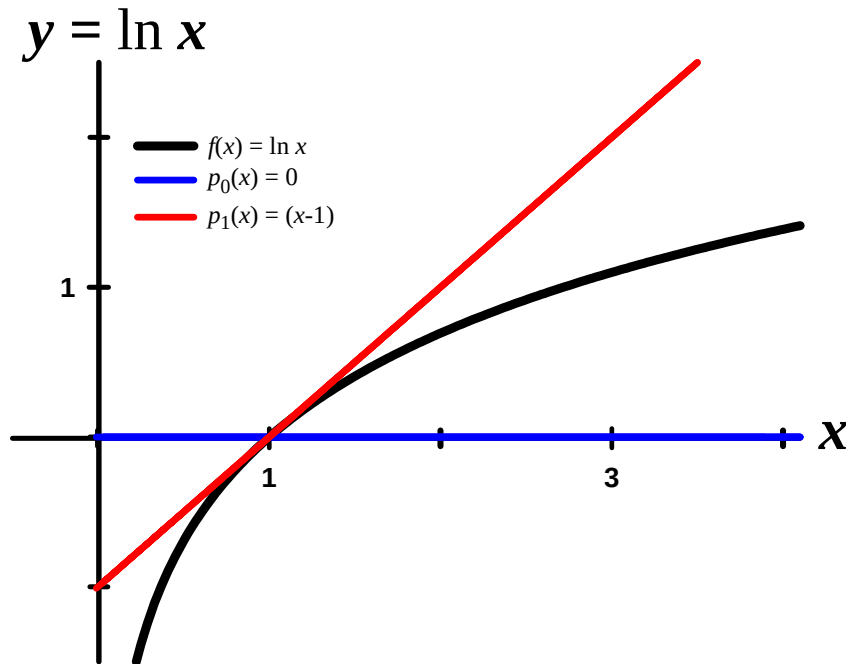
$$p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$



Sequence of Taylor Polynomials

$$p_0(x) = 0$$

$$p_1(x) = (x-1)$$

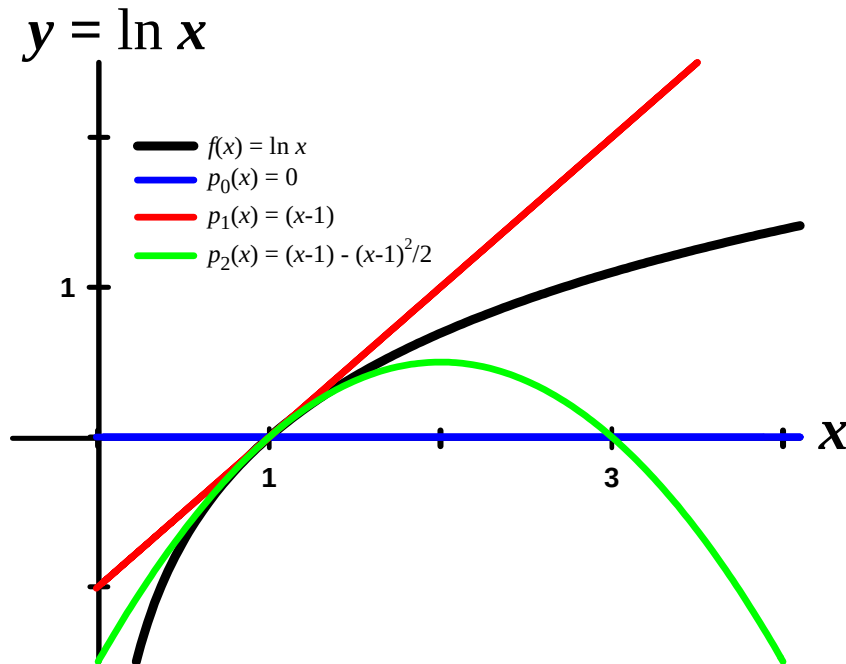
$$p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Second Degree Polynomial Approximation

$$p_2(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$



Sequence of Taylor Polynomials

$$p_0(x) = 0$$

$$p_1(x) = (x-1)$$

$$p_2(x) = (x-1) - \frac{1}{2}(x-1)^2$$

Higher Degree Polynomial Approximation

IDEA: Find a polynomial $p(x)$ that matches $f(x)$ and its first n derivatives at $x = a$.

Higher Degree Polynomial Approximation

IDEA: Find a polynomial $p(x)$ that matches $f(x)$ and its first n derivatives at $x = a$.

Having $n + 1$ things to match means we need a polynomial with $n + 1$ coefficients

Higher Degree Polynomial Approximation

IDEA: Find a polynomial $p(x)$ that matches $f(x)$ and its first n derivatives at $x = a$.

Having $n + 1$ things to match means we need a polynomial with $n + 1$ coefficients

i.e., a polynomial of degree n :

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

$$\frac{d^2 p(x)}{dx^2} = 2c_2 + 3 \cdot 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

$$\frac{d^2 p(x)}{dx^2} = 2c_2 + 3 \cdot 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$

$$\frac{d^3 p(x)}{dx^3} = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

$$\frac{d^2 p(x)}{dx^2} = 2c_2 + 3 \cdot 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$

$$\frac{d^3 p(x)}{dx^3} = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$

$\vdots$

$$\frac{d^n p(x)}{dx^n} = n(n-1)(n-2) \cdots 2c_n$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1}$$

$$\frac{d^2 p(x)}{dx^2} = 2c_2 + 3 \cdot 2c_3(x-a) + \cdots + n(n-1)c_n(x-a)^{n-2}$$

$$\frac{d^3 p(x)}{dx^3} = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$

$$\vdots \quad n!$$

$$\frac{d^n p(x)}{dx^n} = n(n-1)(n-2) \cdots 2 c_n$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

$$\frac{dp(x)}{dx} = c_1 + 2c_2 (x - a) + 3c_3 (x - a)^2 + \cdots + nc_n (x - a)^{n-1}$$

$$\frac{d^2 p(x)}{dx^2} = 2c_2 + 3 \cdot 2c_3 (x - a) + \cdots + n(n-1)c_n (x - a)^{n-2}$$

$$\frac{d^3 p(x)}{dx^3} = 3 \cdot 2c_3 + \cdots + n(n-1)(n-2)c_n (x - a)^{n-3}$$

$\vdots$

$$\frac{d^n p(x)}{dx^n} = n!c_n$$

Matching with $f(x)$ at $x = a$
requires *setting* $x = a$
and *equating* the results
to the derivatives of $f(x)$

Higher Degree Polynomial Approximation

$$p(a) = c_0$$

$$\left. \frac{dp(x)}{dx} \right|_{x=a} = c_1$$

$$\left. \frac{d^2 p(x)}{dx^2} \right|_{x=a} = 2c_2$$

$$\left. \frac{d^3 p(x)}{dx^3} \right|_{x=a} = 3 \cdot 2c_3$$

$\vdots$

$$\left. \frac{d^n p(x)}{dx^n} \right|_{x=a} = n!c_n$$

Matching with $f(x)$ at $x = a$
requires *setting* $x = a$
and *equating* the results
to the derivatives of $f(x)$

Higher Degree Polynomial Approximation

$$p(a) = c_0 = f(a)$$

$$\left. \frac{dp(x)}{dx} \right|_{x=a} = c_1 = \left. \frac{df(x)}{dx} \right|_{x=a}$$

$$\left. \frac{d^2 p(x)}{dx^2} \right|_{x=a} = 2c_2 = \left. \frac{d^2 f(x)}{dx^2} \right|_{x=a}$$

$$\left. \frac{d^3 p(x)}{dx^3} \right|_{x=a} = 3 \cdot 2c_3 = \left. \frac{d^3 f(x)}{dx^3} \right|_{x=a}$$

$\vdots$

$$\left. \frac{d^n p(x)}{dx^n} \right|_{x=a} = n!c_n = \left. \frac{d^n f(x)}{dx^n} \right|_{x=a}$$

Matching with $f(x)$ at $x = a$
requires *setting* $x = a$
and *equating* the results
to the derivatives of $f(x)$

Higher Degree Polynomial Approximation

$$p(a) = c_0 = f(a)$$

$$\left. \frac{dp(x)}{dx} \right|_{x=a} = c_1 = \left. \frac{df(x)}{dx} \right|_{x=a}$$

$$\left. \frac{d^2 p(x)}{dx^2} \right|_{x=a} = 2c_2 = \left. \frac{d^2 f(x)}{dx^2} \right|_{x=a}$$

$$\left. \frac{d^3 p(x)}{dx^3} \right|_{x=a} = 3 \cdot 2c_3 = \left. \frac{d^3 f(x)}{dx^3} \right|_{x=a}$$

$\vdots$

$$\left. \frac{d^n p(x)}{dx^n} \right|_{x=a} = n!c_n = \left. \frac{d^n f(x)}{dx^n} \right|_{x=a}$$

This formula tells us how to calculate the coefficients c_n of the approximating polynomial

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

$$c_0 = f(a), \quad c_n = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=a} \quad \text{for } n = 1, 2, \dots, n$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

$$c_0 = f(a), \quad c_n = \frac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x=a} \quad \text{for } n = 1, 2, \dots, n$$

A shorthand notation :

$$f^{(n)}(a) = \begin{cases} f(a) & \text{for } n = 0 \\ \left. \frac{d^n f(x)}{dx^n} \right|_{x=a} & \text{for } n = 1, 2, \dots, n \end{cases}$$

Higher Degree Polynomial Approximation

$$p(x) = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \cdots + c_n (x - a)^n$$

$$c_n = \frac{1}{n!} f^{(n)}(a) \quad \text{for } n = 0, 1, 2, \dots, n$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

The *Taylor polynomial* of degree n of $f(x)$ at $x = a$.

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = 1 + x$$

$$f^{(2)}(0) = e^0 = 1 \Rightarrow f^{(2)}(0) = 1 \Rightarrow p_2(x) = 1 + x + (1/2)x^2$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = 1 + x$$

$$f^{(2)}(0) = e^0 \Rightarrow f^{(2)}(0) = 1 \Rightarrow p_2(x) = 1 + x + (1/2)x^2$$

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = 1$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = 1 + x$$

$$f^{(2)}(0) = e^0 \Rightarrow f^{(2)}(0) = 1 \Rightarrow p_2(x) = 1 + x + (1/2)x^2$$

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = 1 \Rightarrow p_3(x) = 1 + x + (1/2)x^2 + (1/3!)x^3$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = 1 + x$$

$$f^{(2)}(x) = e^x \Rightarrow f^{(2)}(0) = 1 \Rightarrow p_2(x) = 1 + x + (1/2)x^2$$

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = 1 \Rightarrow p_3(x) = 1 + x + (1/2)x^2 + (1/3!)x^3$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1 \Rightarrow p_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = e^x \quad \text{at } a = 0$$

$$f^{(1)}(x) = e^x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = 1 + x$$

$$f^{(2)}(x) = e^x \Rightarrow f^{(2)}(0) = 1 \Rightarrow p_2(x) = 1 + x + (1/2)x^2$$

$$f^{(3)}(x) = e^x \Rightarrow f^{(3)}(0) = 1 \Rightarrow p_3(x) = 1 + x + (1/2)x^2 + (1/3!)x^3$$

$$f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1 \Rightarrow p_n(x) = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$$

n^{th} degree Taylor polynomial at $x = 0$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \quad \Rightarrow \quad f^{(1)}(1) = 1 \quad \Rightarrow \quad p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \quad \Rightarrow \quad f^{(2)}(1) = -1 \quad \Rightarrow \quad p_2(x) = (x-1) - (1/2)(x-1)^2$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \quad \Rightarrow \quad f^{(1)}(1) = 1 \quad \Rightarrow \quad p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \Rightarrow f^{(2)}(1) = -1 \Rightarrow \quad p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \Rightarrow f^{(3)}(1) = 2 \Rightarrow$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \quad \Rightarrow \quad f^{(1)}(1) = 1 \quad \Rightarrow \quad p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \quad \Rightarrow \quad f^{(2)}(1) = -1 \quad \Rightarrow \quad p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \quad \Rightarrow \quad f^{(3)}(1) = 2 \quad \Rightarrow \quad p_3(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \Rightarrow f^{(1)}(1) = 1 \Rightarrow p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \Rightarrow f^{(2)}(1) = -1 \Rightarrow p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \Rightarrow f^{(3)}(1) = 2 \Rightarrow p_3(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3$$

$$f^{(4)}(x) = -3!/x^4 \Rightarrow f^{(4)}(1) = -3! \Rightarrow$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \Rightarrow f^{(1)}(1) = 1 \Rightarrow p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \Rightarrow f^{(2)}(1) = -1 \Rightarrow p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \Rightarrow f^{(3)}(1) = 2 \Rightarrow p_3(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3$$

$$f^{(4)}(x) = -3!/x^4 \Rightarrow f^{(4)}(1) = -3! \Rightarrow$$

$$p_4(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3 - (1/4)(x-1)^4$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \Rightarrow f^{(1)}(1) = 1 \Rightarrow p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \Rightarrow f^{(2)}(1) = -1 \Rightarrow p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \Rightarrow f^{(3)}(1) = 2 \Rightarrow p_3(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3$$

$$f^{(4)}(x) = -3!/x^4 \Rightarrow f^{(4)}(1) = -3! \Rightarrow$$

$$p_4(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3 - (1/4)(x-1)^4$$

alternating signs
reciprocals of integers

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \ln x \quad \text{at } a = 1$$

$$f^{(1)}(x) = 1/x \Rightarrow f^{(1)}(1) = 1 \Rightarrow p_1(x) = (x-1)$$

$$f^{(2)}(x) = -1/x^2 \Rightarrow f^{(2)}(1) = -1 \Rightarrow p_2(x) = (x-1) - (1/2)(x-1)^2$$

$$f^{(3)}(x) = 2/x^3 \Rightarrow f^{(3)}(1) = 2 \Rightarrow p_3(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3$$

$$f^{(4)}(x) = -3!/x^4 \Rightarrow f^{(4)}(1) = -3! \Rightarrow$$

$$p_4(x) = (x-1) - (1/2)(x-1)^2 + (1/3)(x-1)^3 - (1/4)(x-1)^4$$

$$p_n(x) = (x-1) - \frac{1}{2}(x-1)^2 + \cdots + \frac{(-1)^{n-1}}{n}(x-1)^n$$

n^{th} degree Taylor polynomial at $x = 0$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$f^{(1)}(x) = \cos x \quad \Rightarrow \quad f^{(1)}(0) = 1 \quad \Rightarrow \quad p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \quad \Rightarrow \quad f^{(2)}(0) = 0 \quad \Rightarrow \quad p_2(x) = x$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0 \Rightarrow p_2(x) = x$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1 \Rightarrow p_3(x) = x - (1/3!)x^3$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0 \Rightarrow p_2(x) = x$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1 \Rightarrow p_3(x) = x - (1/3!)x^3$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0 \Rightarrow p_4(x) = x - (1/3!)x^3$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0 \Rightarrow p_2(x) = x$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1 \Rightarrow p_3(x) = x - (1/3!)x^3$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0 \Rightarrow p_4(x) = x - (1/3!)x^3$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1 \Rightarrow p_5(x) = x - (1/3!)x^3 + (1/5!)x^5$$

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \quad \text{at } a = 0$$

$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0 \Rightarrow p_2(x) = x$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1 \Rightarrow p_3(x) = x - (1/3!)x^3$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0 \Rightarrow p_4(x) = x - (1/3!)x^3$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1 \Rightarrow p_5(x) = x - (1/3!)x^3 + (1/5!)x^5$$

even powers are missing
odd powers alternate sign

Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$f^{(1)}(x) = \cos x \Rightarrow f^{(1)}(0) = 1 \Rightarrow p_1(x) = x$$

$$f^{(2)}(x) = -\sin x \Rightarrow f^{(2)}(0) = 0 \Rightarrow p_2(x) = x$$

$$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -1 \Rightarrow p_3(x) = x - (1/3!)x^3$$

$$f^{(4)}(x) = \sin x \Rightarrow f^{(4)}(0) = 0 \Rightarrow p_4(x) = x - (1/3!)x^3$$

$$f^{(5)}(x) = \cos x \Rightarrow f^{(5)}(0) = 1 \Rightarrow p_5(x) = p_3(x) = x - (1/3!)x^3 + (1/5!)x^5$$

$$p_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

n^{th} degree Taylor polynomial at $x = 0$

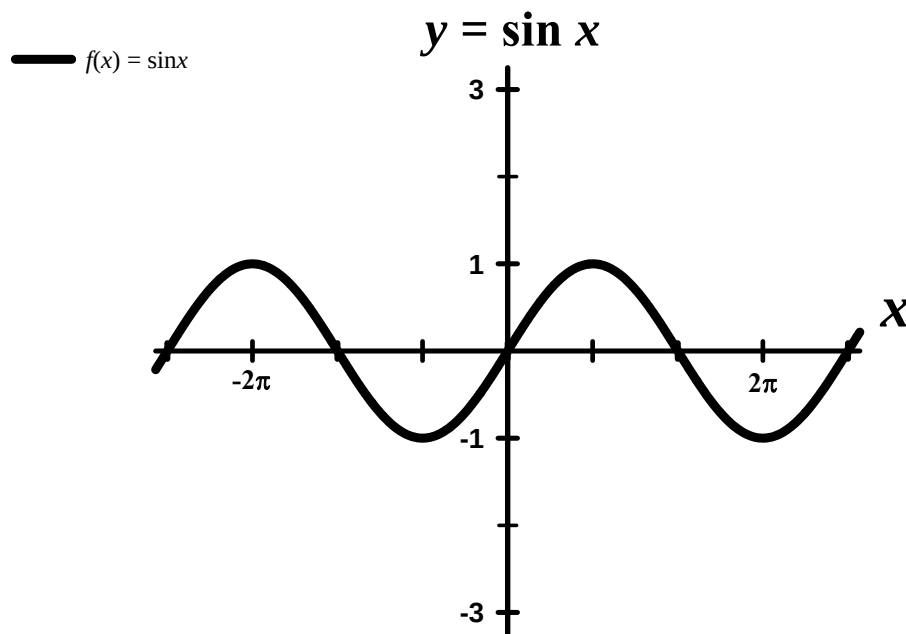
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

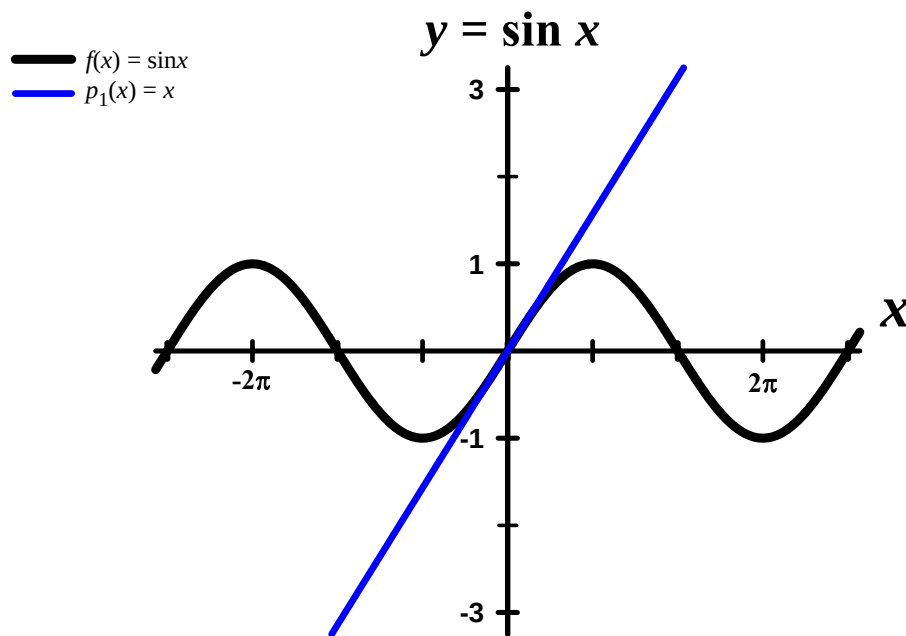


Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$
$$p_n(x) = \boxed{x} - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



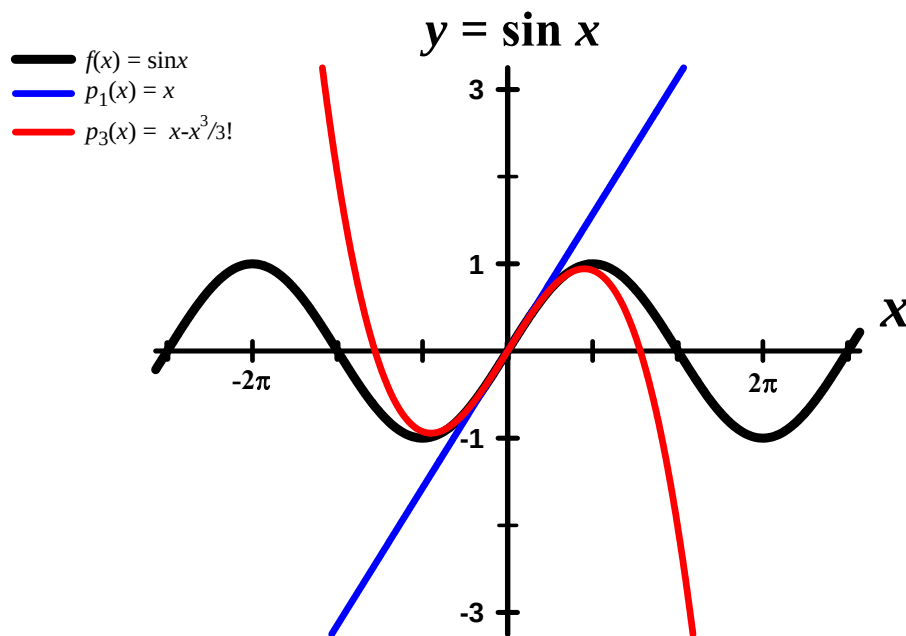
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = \boxed{x - \frac{1}{3!}x^3} + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$



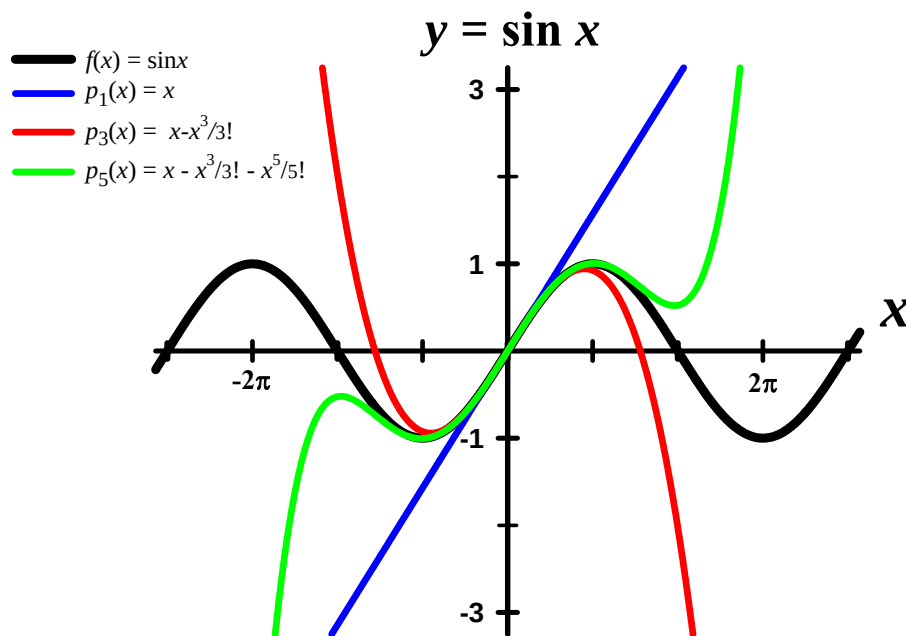
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$



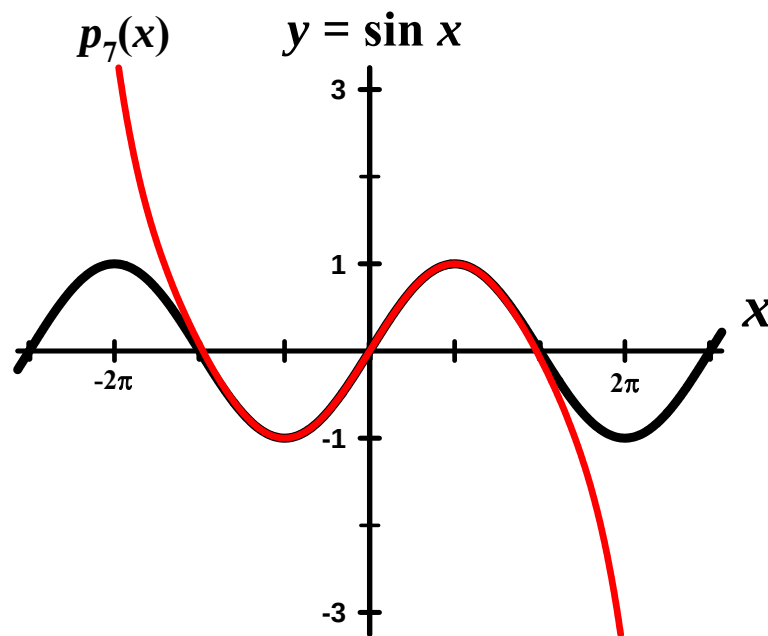
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



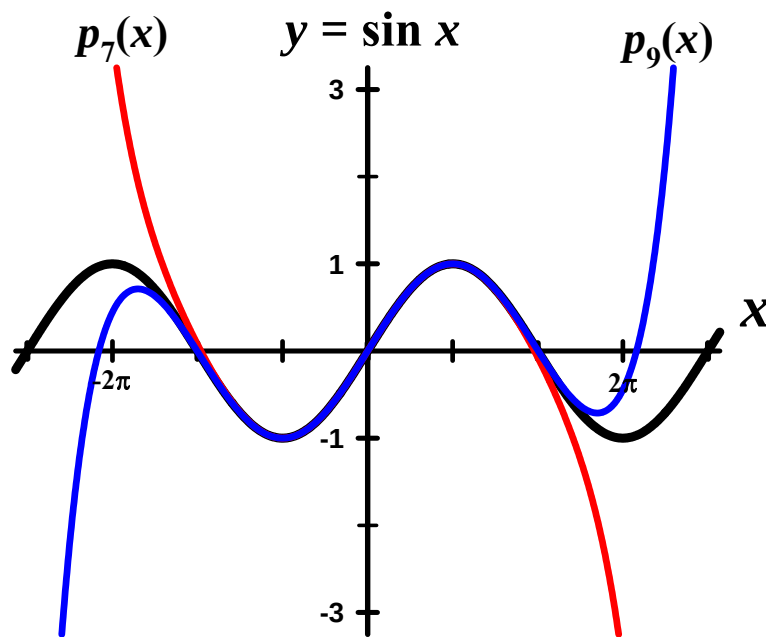
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



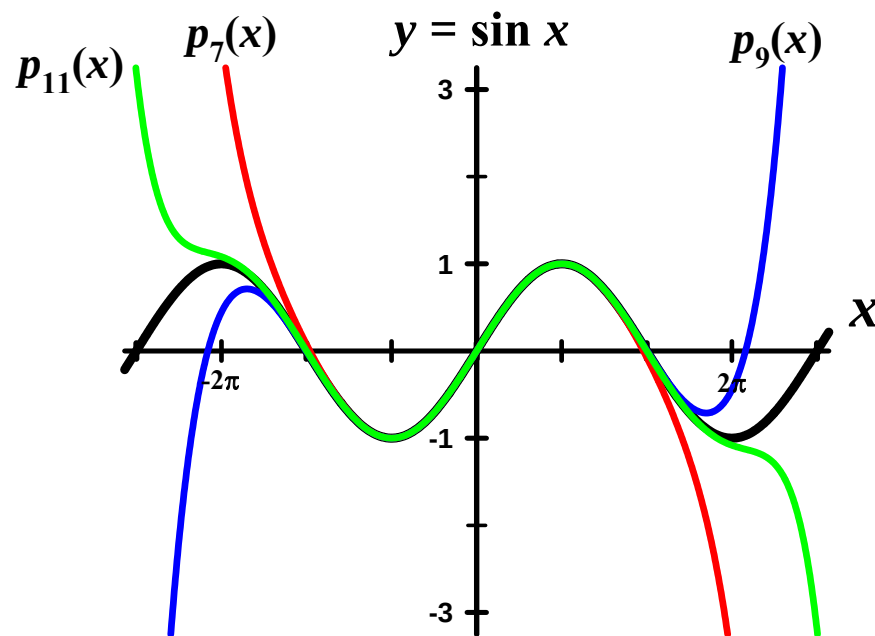
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



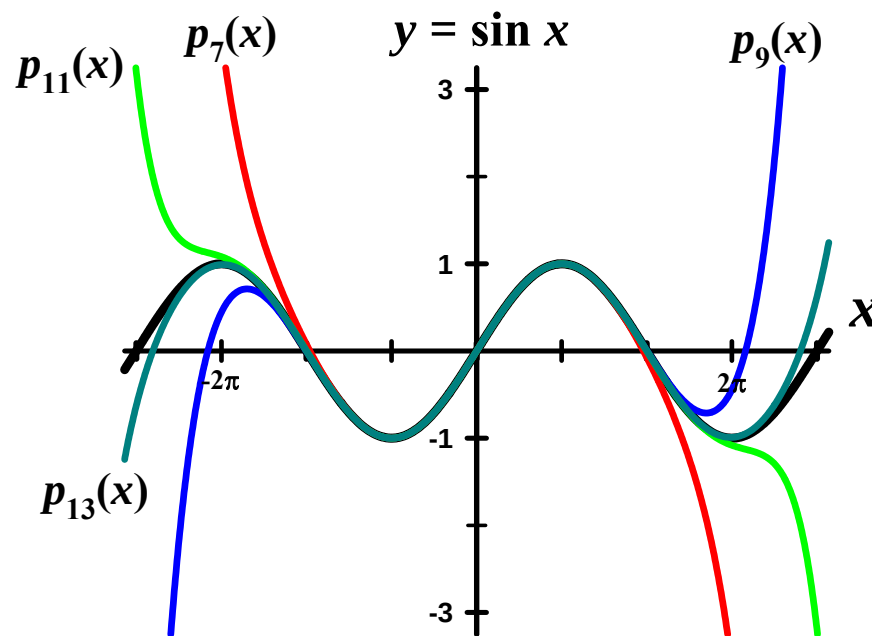
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



It seems that

$$\lim_{n \rightarrow +\infty} p_n(x) = \sin x$$

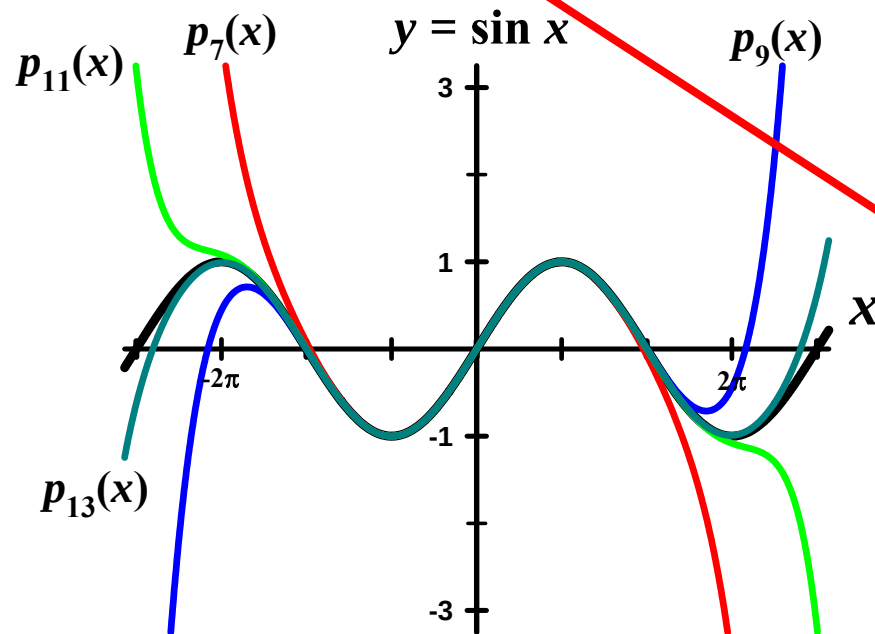
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



It seems that

$$\lim_{n \rightarrow +\infty} p_n(x) = \sin x$$

Note: $p_n(x)$ is a
sequence of partial
sums

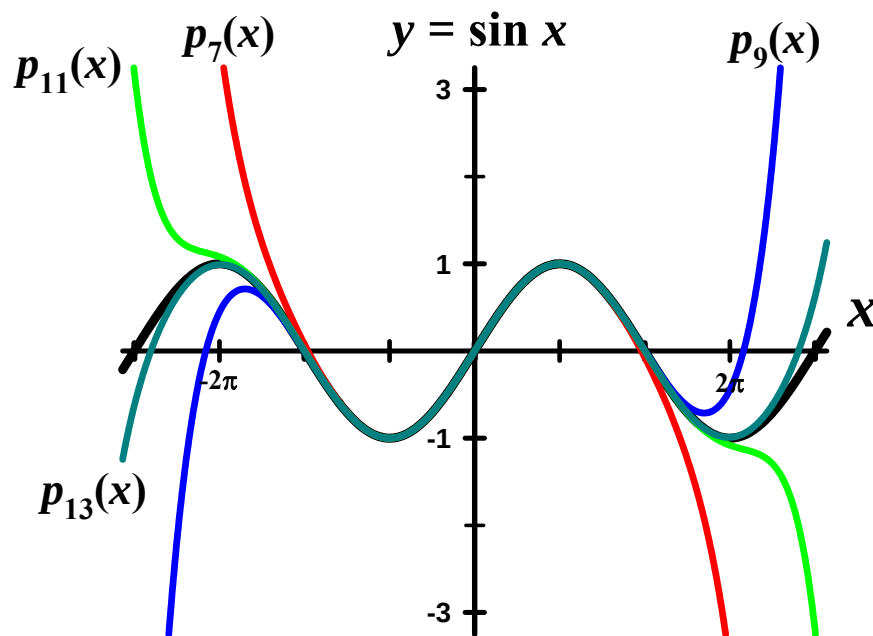
Higher Degree Polynomial Approximation

$$p_n(x) = f(a) + f^{(1)}(a)(x-a) + \frac{1}{2} f^{(2)}(a)(x-a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x-a)^n$$

Examples

$$f(x) = \sin x \text{ at } a = 0$$

$$p_n(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$



It seems that

$$\lim_{n \rightarrow +\infty} p_n(x) = \sin x$$

Note: this is an
infinite series

10.3 Taylor Series

The sequence of Taylor polynomials of a function $f(x)$

$$p_n(x) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(a)(x-a)^i$$

is the sequence of partial sums of the *Taylor series*

$$\sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(a)(x-a)^i.$$

10.3 Taylor Series

The sequence of Taylor polynomials of a function $f(x)$

$$p_n(x) = \sum_{i=0}^n \frac{1}{i!} f^{(i)}(a)(x-a)^i$$

is the sequence of partial sums of the *Taylor series*

$$\sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(a)(x-a)^i.$$

If R is the radius of convergence of this Taylor series, then

$$\sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(a)(x-a)^i = f(x)$$

on the interval $|x-a| < R$.

Examples

We saw in two previous examples that

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has radius of convergence $R = +\infty$.

Examples

We saw in two previous examples that

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has radius of convergence $R = +\infty$.

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is the power series of $f(x) = e^x$ at $a = 0$.

Examples

We saw in two previous examples that

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ has radius of convergence $R = +\infty$.

$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is the power series of $f(x) = e^x$ at $a = 0$.

We conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x \quad \text{for all } x.$$

Examples

We saw in two previous examples that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ has radius of convergence } R = +\infty.$$

Examples

We saw in two previous examples that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ has radius of convergence } R = +\infty.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ is the power series of } f(x) = \sin x \text{ at } a = 0.$$

Examples

We saw in two previous examples that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ has radius of convergence } R = +\infty.$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \text{ is the power series of } f(x) = \sin x \text{ at } a = 0.$$

We conclude that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x \text{ for all } x.$$

Examples

We saw in two previous examples that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{has radius of convergence} \quad R = 1.$$

Examples

We saw in two previous examples that

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$ has radius of convergence $R = 1$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$ is the power series of $f(x) = \ln x$ at $a = 1$.

Examples

We saw in two previous examples that

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$ has radius of convergence $R = 1$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n$ is the power series of $f(x) = \ln x$ at $a = 1$.

We conclude that

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \ln x$ on the interval $0 < x < 2$.

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

The Binomial Series

$$(1+x)^p$$

p is any constant (not necessarily an integer)

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots$$

p is any constant (not necessarily an integer)

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots \quad \text{for } -1 < x < 1$$

p is any constant (not necessarily an integer)

Some Basic Power Series Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{for all } x$$

$$\ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \text{for } 0 < x < 2$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \text{for all } x$$

The Binomial Series

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots \quad \text{for } -1 < x < 1$$

p is any constant (not necessarily an integer)

$$(1+x)^p = 1 + \sum_{n=1}^{\infty} \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$$