

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

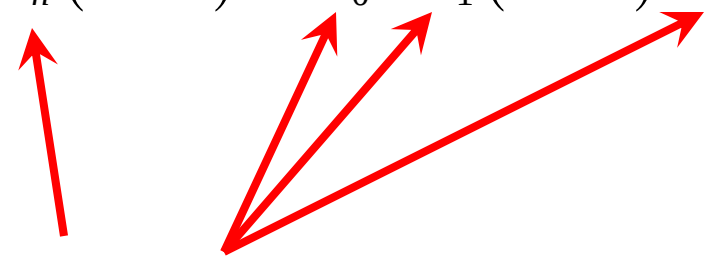
Thursday, November 8

<http://math.arizona.edu/~cushing/250a.html>

9.5 Power Series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

the coefficients



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$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$$

the coefficients

the center

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Examples

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \cdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = (x-1) - \frac{1}{2} (x-1)^2 + \frac{1}{3} (x-1)^3 - \frac{1}{4} (x-1)^4 + \cdots$$

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Examples

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Note “missing” terms

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots$$

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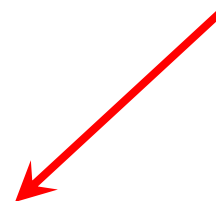
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Center $a = 1$



9.5 Power Series

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Basic Questions

- For which values of x does a power series converge ?
- To what does a power series converge ?
... by this is meant: is there an alternative, non-power series formula ?

Note: a power series always converges to c_0 at its center $x = a$.

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Basic Fact

A power series converges *on an interval* $|x-a| < R$

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and diverges for $|x-a| > R$

for either some number $R \geq 0$ or for $R = +\infty$.

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Notes

- R is called the *radius of convergence*

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Notes

- R is called the *radius of convergence*
- The series might or might not converge at the endpoints $a \pm R$ of the *convergence interval*.

Examples

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x^n$$

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A geometric series with ratio $r = x/2$.

Examples

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converges for $\left|\frac{x}{2}\right| < 1$

diverges for $\left|\frac{x}{2}\right| > 1$

Examples

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$$\begin{array}{ll} \text{converges for } \left|\frac{x}{2}\right| < 1 & \Rightarrow \text{converges for } |x| < 2 \\ \text{diverges for } \left|\frac{x}{2}\right| > 1 & \text{diverges for } |x| > 2 \end{array}$$

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Radius of convergence $R = 2$

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Radius of convergence $R = 2$

In fact ...

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x^n = \frac{1}{1 - \frac{x}{2}}$$

Examples

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$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x^n = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2 - x}$$

We have “summed the series”.

Examples

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n x^n = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \dots$$

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It is not often possible to do this.
(We’ll see some methods next chapter.)

Examples

$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n (x-3)^n$$

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$$\sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n (x-3)^n = \sum_{n=0}^{\infty} \left(\frac{x-3}{4}\right)^n$$

Geometric with ratio $r = \frac{x-3}{4}$

Examples

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$$\text{for } \left| \frac{x-3}{4} \right| < 1$$

Examples

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$$\text{for } \left|\frac{x-3}{4}\right| < 1 \quad \text{or} \quad |x-3| < 4$$

Radius of convergence $R = 4$

Here's a way to calculate the radius of convergence
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Apply the ratio test to the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|c_{n+1}| |x-a|^{n+1}}{|c_n| |x-a|^n}$$

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$$\text{If } \lim_{n \rightarrow +\infty} \frac{|c_{n+1}|}{|c_n|} = K, \text{ then } \lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = K |x-a|.$$

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Apply ratio test

$$K |x-a| < 1 \Rightarrow \text{convergence}$$

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$$K |x-a| < 1 \Rightarrow \text{convergence}$$

$$K |x-a| > 1 \Rightarrow \text{divergence}$$

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$$|x-a| < 1/K \Rightarrow \text{convergence}$$

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Apply ratio test

$$|x-a| < 1/K \Rightarrow \text{convergence}$$

$$|x-a| > 1/K \Rightarrow \text{divergence}$$

Radius of convergence is $R = 1/K$!

Here's a way to calculate the radius of convergence for
non-geometric series ...

Radius of Convergence (Theorem 9.10, text)

If $\lim_{n \rightarrow +\infty} \frac{|c_{n+1}|}{|c_n|} = K$, then the radius of convergence of $\sum c_n (x - a)^n$
is $R = 1 / K$.

Note:

If $K = \infty$, then $R = 0$.

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Examples

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \dots$$

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$\Rightarrow$ radius of convergence is $R = \infty$

This power series converges *for all* x

Examples

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$\Rightarrow$ the radius of convergence is $R = 1/K = 1$

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$\Rightarrow$ the radius of convergence is $R = 1/K = 1$

This power series converges on the interval

$$|x-1| < 1 \quad \text{or} \quad 0 < x < 2$$

Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots$$

“missing” terms have coefficients = 0.

This implies we cannot calculate $|c_{n+1}|/|c_n|$ when $c_n = 0$
(which is every other ratio !)

Examples

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Apply ratio test directly ...

$$\frac{|a_{n+1}|}{|a_n|}$$

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$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-1)^{n+1}}{(2(n+1)+1)!} x^{2(n+1)+1} \right|}{\left| \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right|}$$

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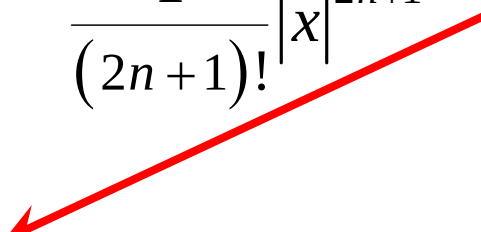
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Examples

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$$\frac{(2n+1)!}{(2n+3)!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n+1)(2n+2)(2n+3)}$$

Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots$$

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$$\frac{(2n+1)!}{(2n+3)!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (2n+1) (2n+2) (2n+3)} = \frac{1}{(2n+2)(2n+3)}$$


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Apply ratio test directly ...

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1}{(2n+2)(2n+3)} |x|^2$$

Examples

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \dots$$

Apply ratio test directly ...

$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow +\infty} \frac{1}{(2n+2)(2n+3)} |x|^2 = 0 < 1$$

for *all* x

Examples

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for *all* x

Ratio test implies ...

series converges for *all* x , that is to say $R = \infty$

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

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Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

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Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (-1)^n$$

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

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Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

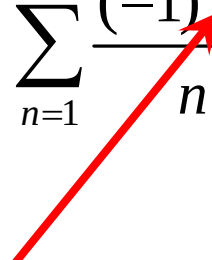
Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{n}$

odd integer power



Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$:
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{-1}{n}$$

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{-1}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$

divergent p -series



Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: series diverges

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

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Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: series diverges

$$\text{At endpoint } x = 2: \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

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$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: series diverges

At endpoint $x = 2$: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

convergent alternating series



Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

At endpoint $x = 0$: series diverges

At endpoint $x = 2$: series converges

Ratio Test doesn't determine what happens at the *endpoints* of the interval of convergence.

Anything goes at the end points ...

Example

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has interval of convergence } 0 < x < 2$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \text{ has domain of convergence } 0 < x \leq 2$$

Chapter 10.1: Taylor Polynomials

Polynomials are easy with which to work (differentiate, integrate, etc.).

For this reason is often convenient to approximate other functions by polynomials.

First Degree Polynomial Approximation

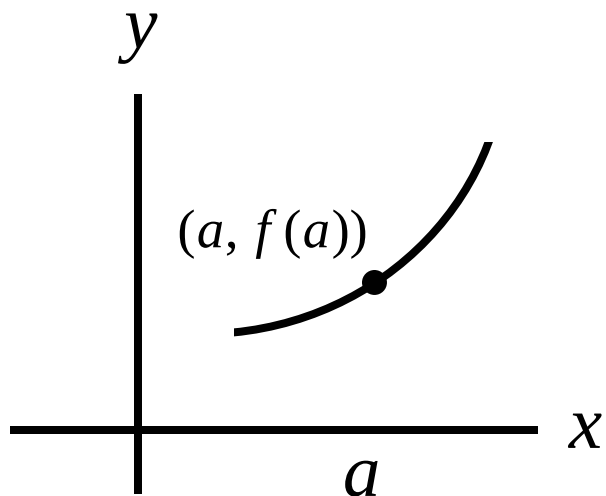
Approximate $y = f(x)$ at $x = a$ by a first degree polynomial.

Geometrically, approximate graph of $f(x)$ by that of its tangent at $x = a$.

First Degree Polynomial Approximation

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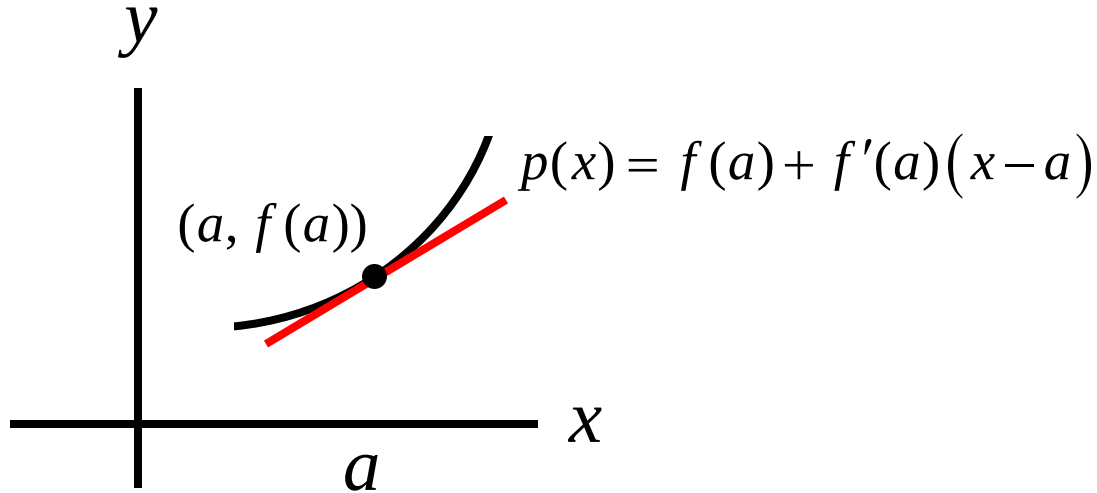
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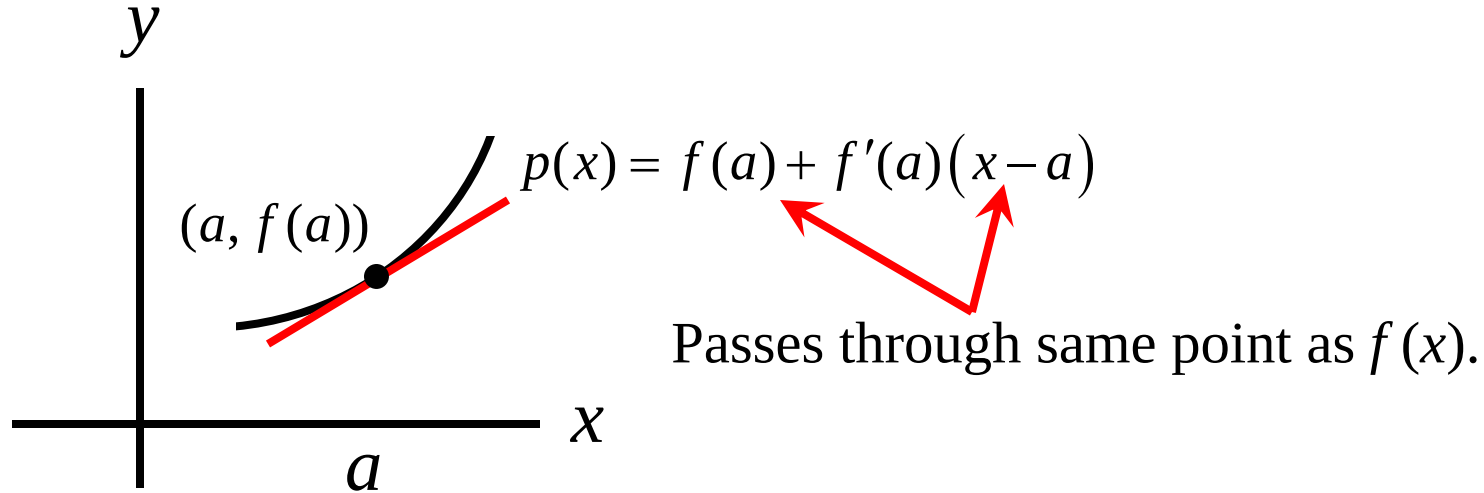
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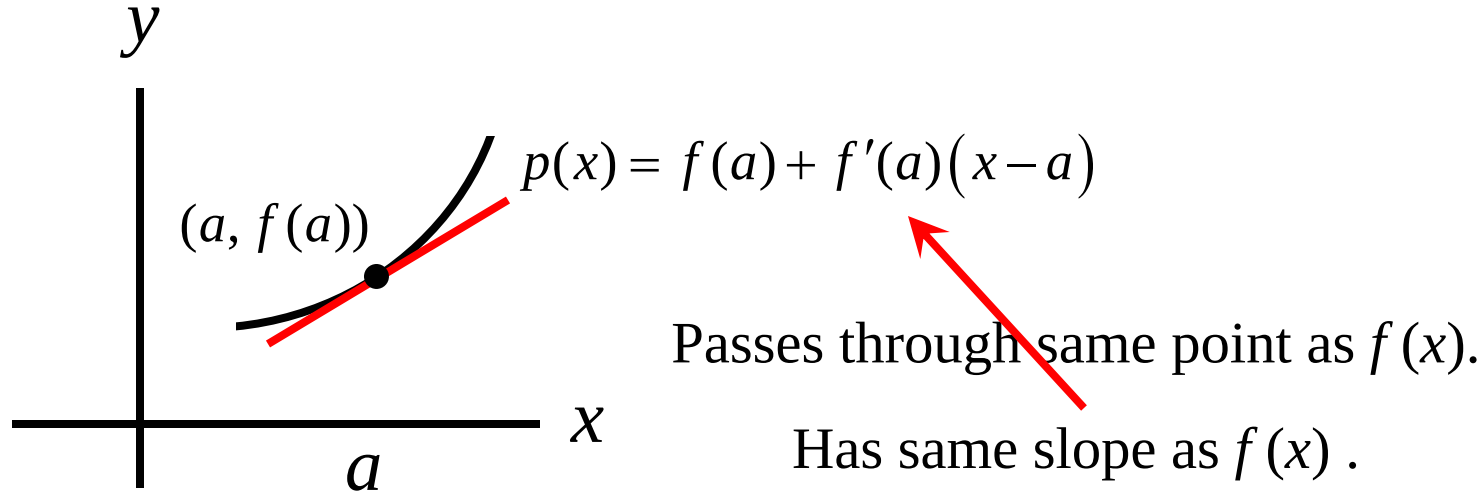
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First Degree Polynomial Approximation

Approximate $y = f(x)$ at $x = a$ by a first degree polynomial.

Geometrically, approximate graph of $f(x)$ by that of its tangent at $x = a$.



First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

First degree Taylor polynomial of $f(x)$ at $x = a$.

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x - a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$p_1(x) = 1 + x$$

First Degree Polynomial Approximation

$$p_1(x) = f(a) + f'(a)(x-a) \approx f(x) \text{ for } x \approx a$$

Examples

$$f(x) = e^x \text{ at } a = 0$$

$$f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$p_1(x) = 1 + x \approx e^x \text{ for } x \approx 0$$