

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, November 6

<http://math.arizona.edu/~cushing/250a.html>

Geometric Series

$$\sum_{n=1}^{\infty} ar^n \quad \begin{cases} \text{converges if } |r| < 1 \\ \text{diverges if } |r| \geq 1 \end{cases}$$

p-Series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$$

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The Comparison Test (Theorem 9.4, text)

Suppose $0 \leq a_n \leq b_n$ for all n .

$$\sum b_n \text{ converges} \Rightarrow \sum a_n \text{ converges}$$

$$\sum a_n \text{ diverges} \Rightarrow \sum b_n \text{ diverges}$$

Limit Comparison Test (Theorem 9.5, text)

Suppose $a_n, b_n > 0$ for all n and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$.

Then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

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What if not all terms in a series are positive ?

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$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \dots$$

An example of an “alternating” series

What if not all terms in a series are positive ?

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$$

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$\sum a_n$ is called *absolutely convergent*

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$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots$$

An alternating p -series

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Convergence does *not* necessarily imply absolute convergence.

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Alternating Series Test (Theorem 9.8, Text)

An alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots$ converges

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$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

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This example shows that

although “absolute convergence implies convergence” is true,
the *converse* “convergence implies absolute convergences” is *not* true

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$$0 < b_{n+1} < b_n, \lim_{b \rightarrow +\infty} b_n = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} b_n = S \quad \text{converges}$$

Error Bound (Theorem 9.9, Text)

$$S_n = n^{\text{th}} \text{ partial sum} \Rightarrow |S - S_n| < b_{n+1}$$

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$\underbrace{\hspace{1.5cm}}_{S_2}$

accurate to within b_3

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$\underbrace{\hspace{10em}}_{S_4}$

accurate to within b_5

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$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = \underbrace{b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots}_{S_5}$$

accurate to within b_6

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$$S_n = n^{\text{th}} \text{ partial sum} \Rightarrow |S - S_n| < b_{n+1}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$\underbrace{\hspace{10em}}_{S_5 = \frac{47}{60}}$

accurate to within $1/6$

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$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \frac{47}{60} \pm \frac{1}{6}$$

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$$\frac{47}{60} - \frac{1}{6} \leq \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \leq \frac{47}{60} + \frac{1}{6}$$

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$$0.6167 \leq \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \leq 0.9500$$

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$$1 < L \leq \infty \Rightarrow \sum a_n \text{ diverges}$$

$$L = 1 \Rightarrow \text{no conclusion (series might converge or diverge)}$$

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$\sum |a_n|$ is like a geometric series with ratio $L < 1$

Example

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Recall: $n! =$ product of integers from 1 to n .

Example

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \dots$$

Recall: $n! =$ product of integers from 1 to n .

Example

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Not a geometric series, not a p -series, nor an alternating series.

Example

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$$

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$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!}$$

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$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot (n+1)}$$

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Example

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot n}{1 \cdot 2 \cdot 3 \cdot \dots \cdot n \cdot (n+1)} = \frac{1}{n+1}$$

$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow +\infty} \frac{1}{n+1} = 0$$

Limit exists

Example

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \dots$$

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$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow +\infty} \frac{1}{n+1} = 0 = L < 1$$

Ratio Test implies convergence

Example

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

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In both cases

$$\lim_{n \rightarrow +\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow +\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n \rightarrow +\infty} \frac{n}{n+1}$$

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In both cases

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Ratio Test does not apply.