

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, October 30

<http://math.arizona.edu/~cushing/250a.html>

## Formal Definition

A sequence  $x_n$  *converges* if there is a number  $L$  (called the *limit* of the sequence) for which the following is true :  
for *any* small, positive number  $\varepsilon$  there is a number  $N$  so that

$$|x_n - L| < \varepsilon \text{ for all } n > N.$$

The sequence gets and remains as close to  $L$  as you want,  
provided you go far enough out the sequence.

### Example

Prove that  $x_n = 0.5^n$  converges to  $L = 0$ .

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if

$$\ln 0.5^n < \ln \varepsilon$$

$$n \ln 0.5 < \ln \varepsilon$$

$$n > \ln \varepsilon / \ln 0.5 :$$



because  $\ln 0.5 < 0$

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For example,  $|x_n| < 0.1$  provided  $n > \ln 0.1 / \ln 0.5 \approx 3.32$ .

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For example,  $|x_n| < 0.1$  provided  $n > \ln 0.1 / \ln 0.5 \approx 3.32$ .

$|x_n| < 0.01$  provided  $n > \ln 0.01 / \ln 0.5 \approx 6.64$ .

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For example,  $|x_n| < 0.1$  provided  $n > \ln 0.1 / \ln 0.5 \approx 3.32$ .

$|x_n| < 0.01$  provided  $n > \ln 0.01 / \ln 0.5 \approx 6.64$ .

Similarly, we can show

$x_n = r^n$  converges to 0 if  $|r| < 1$

$x_n = r^n$  diverges if  $|r| \geq 1$

## Formal Definition

A sequence  $x_n$  *converges* if there is a number  $L$   
(called the *limit* of the sequence) for which the following is true :  
for *any* small, positive number  $\varepsilon$  there is a number  $N$  so that

$$|x_n - L| < \varepsilon \text{ for all } n > N.$$

One difficulty in using this definition to calculate limits in problems is that  
it requires you to know (or have a candidate for) the limit  $L$   
before you can formally verify a sequence converges.

One goal in calculus is  
to learn ways to calculate limits  
other than by means of this formal definition.

## Basic Rules about Limits

Suppose  $x_n$  and  $y_n$  are both convergent sequences.

Then the following sequences are also convergent.

$$z_n = a x_n \text{ for any constant } a$$

$$z_n = x_n + y_n$$

$$z_n = x_n y_n$$

$$z_n = \frac{x_n}{y_n} \text{ provided } \lim_{n \rightarrow +\infty} y_n \neq 0$$

## Basic Rules about Limits

Suppose  $x_n$  and  $y_n$  are both convergent sequences.

In fact, if  $\lim_{n \rightarrow +\infty} x_n = L$  and  $\lim_{n \rightarrow +\infty} y_n = M$ , then

$$\lim_{n \rightarrow +\infty} a x_n = a L \text{ for any constant } a$$

$$\lim_{n \rightarrow +\infty} (x_n + y_n) = L + M$$

$$\lim_{n \rightarrow +\infty} (x_n y_n) = LM$$

$$\lim_{n \rightarrow +\infty} \frac{x_n}{y_n} = \frac{L}{M} \text{ provided } M \neq 0$$

## Example

$$\lim_{n \rightarrow +\infty} \left( \frac{3 - 5 \times 0.1^n}{-1 + 0.7^n} \right)$$

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$$\lim_{n \rightarrow +\infty} \left( \frac{3 - 5 \times 0.1^n}{-1 + 0.7^n} \right) = \frac{\lim_{n \rightarrow +\infty} (3 - 5 \times 0.1^n)}{\lim_{n \rightarrow +\infty} (-1 + 0.7^n)}$$

rule for quotients

## Example

$$\begin{aligned}\lim_{n \rightarrow +\infty} \left( \frac{3 - 5 \times 0.1^n}{-1 + 0.7^n} \right) &= \frac{\lim_{n \rightarrow +\infty} (3 - 5 \times 0.1^n)}{\lim_{n \rightarrow +\infty} (-1 + 0.7^n)} \\ &= \frac{\lim_{n \rightarrow +\infty} 3 + \lim_{n \rightarrow +\infty} (-5 \times 0.1^n)}{\lim_{n \rightarrow +\infty} (-1) + \lim_{n \rightarrow +\infty} 0.7^n}\end{aligned}$$

rule for sums

## Example

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rule for constant multiples

## Example

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we know these limits

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## Example

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## Another Useful Rule

Suppose  $x_n$  converges to  $L$  and  $f(x)$  is continuous at  $L$ .

$$\text{Then } \lim_{n \rightarrow +\infty} f(x_n) = f(L).$$

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$$\text{Then } \lim_{n \rightarrow +\infty} f(x_n) = f(L).$$

### Example

$$\lim_{n \rightarrow +\infty} \cos(0.75^n) = \cos 0 = 1$$

## Example (density regulated growth)

We've seen that a population growing according the the law

$$x_{n+1} = r x_n$$

goes extinct (converges to 0) if  $r < 1$

grows without bound (diverges to  $\infty$ ) if  $r > 1$ .

If we want a model that describes a population that *survives*,  
but does *not* grow without bound,  
then  $r$  *cannot remain constant over time*.

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If we want a model that describes a population that *survives*,  
but does *not* grow without bound,  
then  $r$  *cannot remain constant over time*.

A famous model replaces  $r$  by

$$r \frac{1}{1 + cx_n}$$

$0 < c$  (units = 1 / population number )

The growth factor decreases ( to 0 ) as the population size  $x_n$  increases.

## Example (density regulated growth)

The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

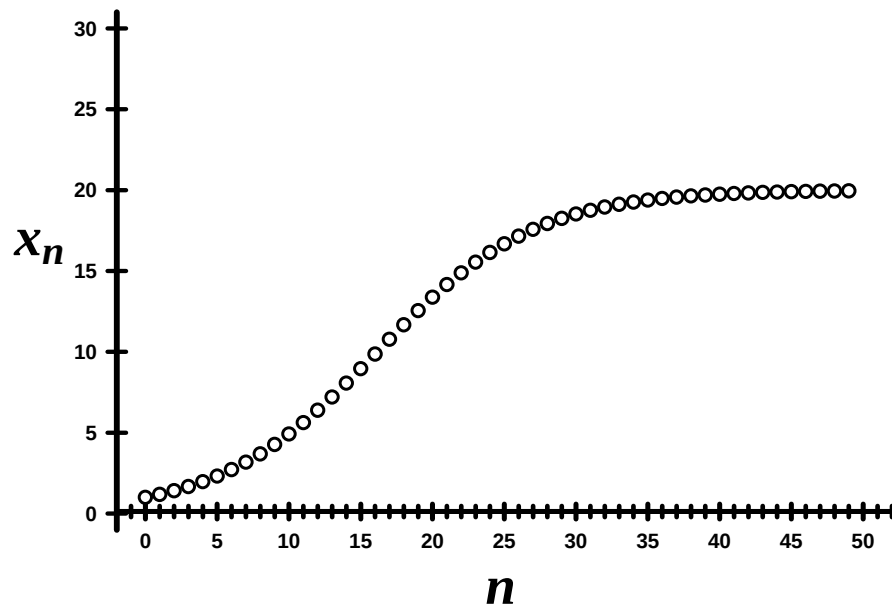
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**Beverton-Holt  $r = 1.2$ ,  $c = 0.01$**



With initial condition  $x_0 = 1$

$x_n$  appears to converge  
(to a limit  $L \approx 20$ )

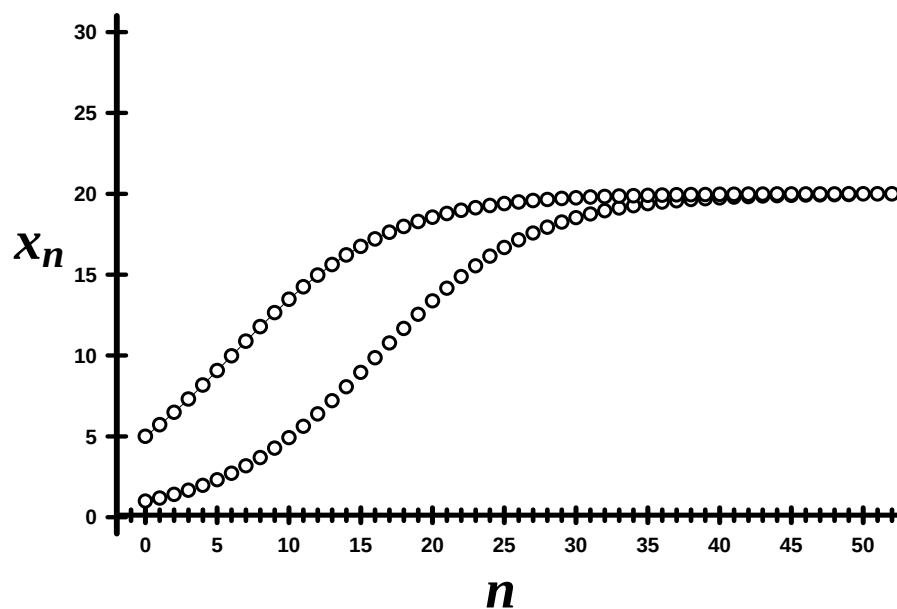
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With initial condition  $x_0 = 5$

$x_n$  appears to converge  
(to the same limit  $L \approx 20$ )

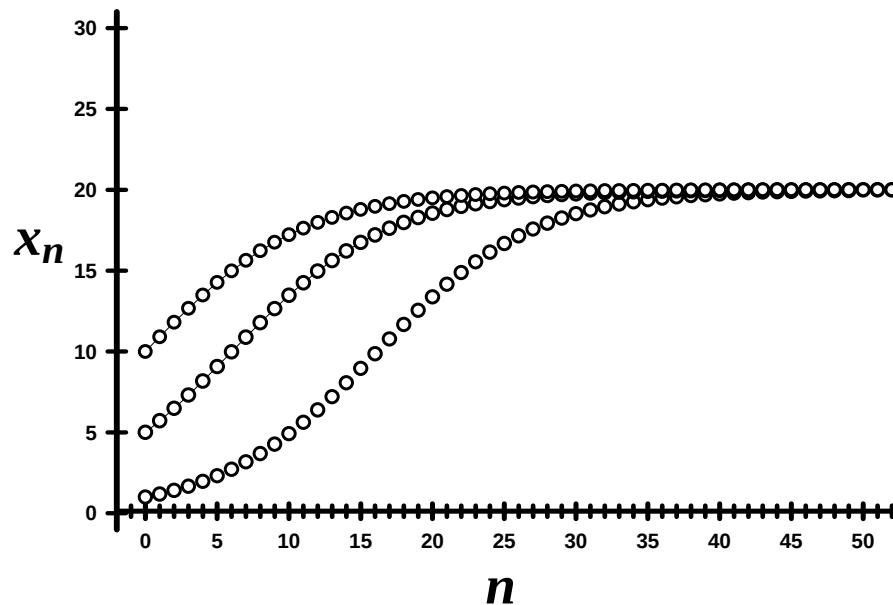
## Example (density regulated growth)

The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

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With initial condition  $x_0 = 10$

$x_n$  appears to converge  
(to the same limit  $L \approx 20$ )

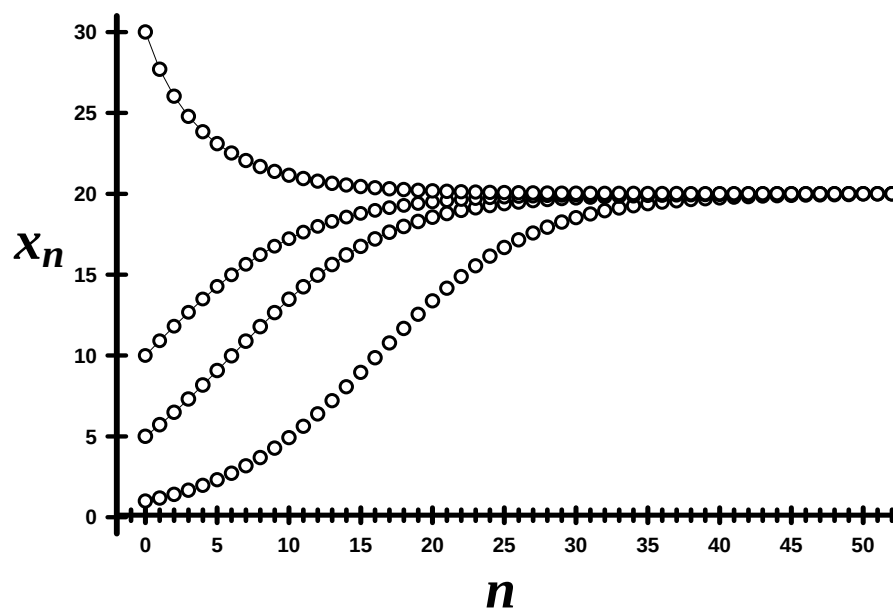
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Question: what do solutions do when  $r > 1$  ?

**Beverton-Holt  $r = 1.2, c = 0.01$**



With initial condition  $x_0 = 30$

$x_n$  appears to converge  
(to the same limit  $L \approx 20$ )

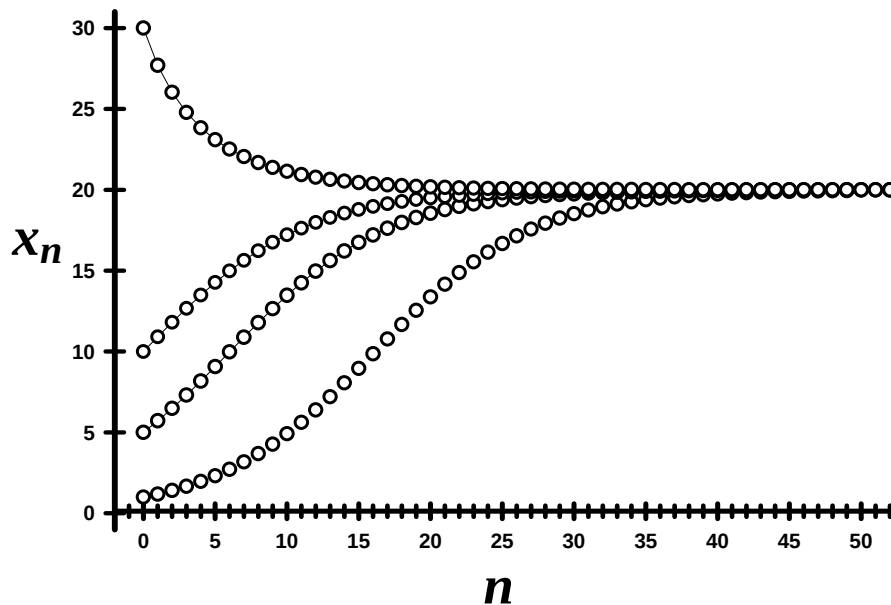
## Example (density regulated growth)

The Beverton-Holt Equation:

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Question: what do solutions do when  $r > 1$  ?

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We conjecture that  
*for all* initial conditions  $x_0 > 0$   
the solution sequence  $x_n$  will converge  
(to the same limit  $L \approx 20$ )

## Example (density regulated growth)

The Beverton-Holt Equation:

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Question: what do solutions do when  $r > 1$  ?

We conjecture that

*for all* initial conditions  $x_0 > 0$

the solution sequence  $x_n$  will *converge*

(to a limit  $L$  that depends on  $r$  and  $c$ )

## Example (density regulated growth)

The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

It turns out there is a solution formula !

$$x_n = g(n)$$

## Example (density regulated growth)

The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

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$$x_n = \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0}$$

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$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0}$$

Use basic rules for taking limits.

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NOTE:  $\lim_{n \rightarrow +\infty} r^{-n}$

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$$\text{NOTE: } \lim_{n \rightarrow +\infty} r^{-n} = \lim_{n \rightarrow +\infty} \left( \frac{1}{r} \right)^n$$

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The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

It turns out there is a solution formula !

$$\frac{1}{r} < 1$$

$$x_n = \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0}$$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0}$$

NOTE:  $\lim_{n \rightarrow +\infty} r^{-n} = \lim_{n \rightarrow +\infty} \left(\frac{1}{r}\right)^n$

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$$\text{NOTE: } \lim_{n \rightarrow +\infty} r^{-n} = \lim_{n \rightarrow +\infty} \left( \frac{1}{r} \right)^n = 0$$

## Example (density regulated growth)

The Beverton-Holt Equation:

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Question: what do solutions do when  $r > 1$  ?

It turns out there is a solution formula !

$$x_n = \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0}$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} x_n &= \lim_{n \rightarrow +\infty} \frac{(r-1)x_0}{[(r-1) - cx_0]r^{-n} + cx_0} \\ &= \frac{(r-1)x_0}{0 + cx_0} \end{aligned}$$

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## Example (density regulated growth)

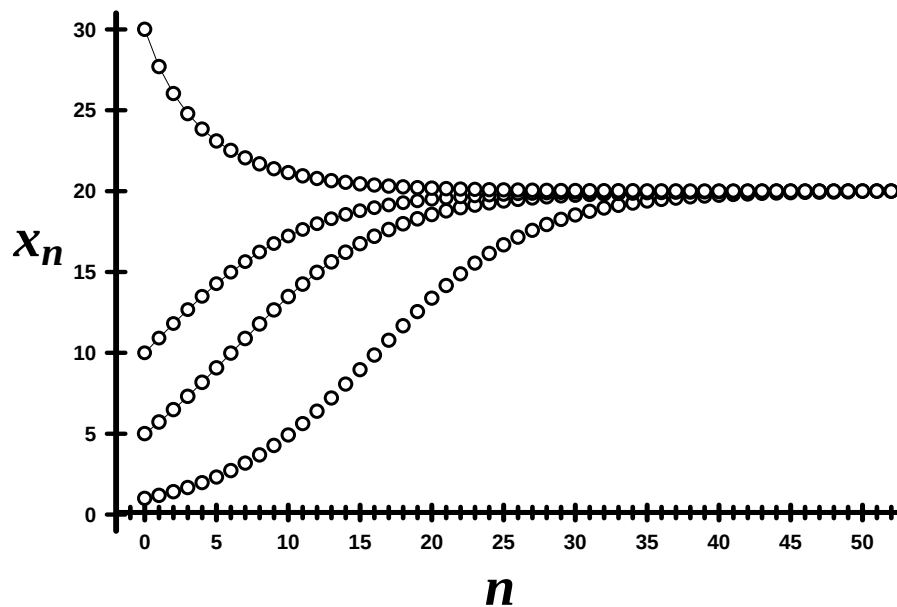
The Beverton-Holt Equation:

$$x_{n+1} = r \frac{1}{1 + cx_n} x_n$$

Question: what do solutions do when  $r > 1$  ?

$$\lim_{n \rightarrow +\infty} x_n = \frac{r-1}{c}$$

**Beverton-Holt  $r = 1.2$ ,  $c = 0.01$**



$$\lim_{n \rightarrow +\infty} x_n = \frac{1.2-1}{0.01} = 20$$

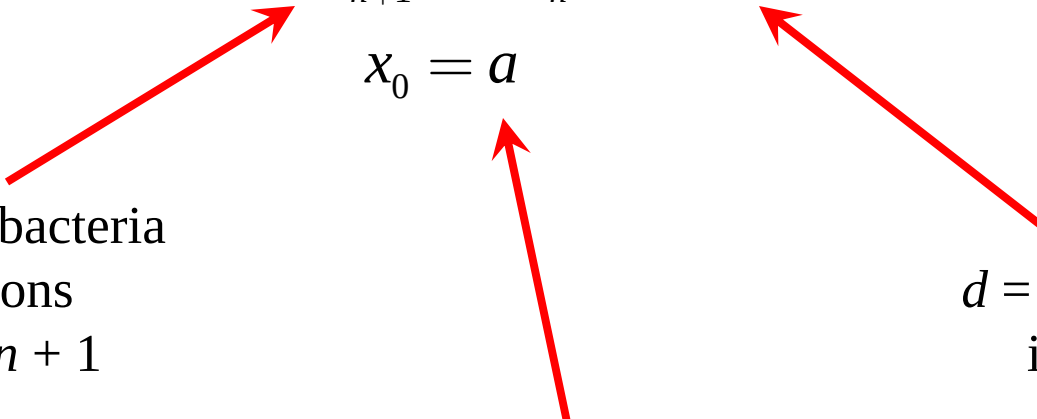
## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

Number of bacteria  
in millions  
at hour  $n + 1$



$d$  = antibiotic dose  
in milligram

$a$  = initial infection  
( millions of *S. aureus* )

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

## Recall Our Conjecture

There is a *threshold* for the dosage  $d$  below which the treatment does *not* cure the infection is and above which it does .

The threshold depends on the initial infection number  $a$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

### Mathematical Formulation of the Conjecture

There is a threshold value of  $d$   
below which the solution  $x_n$  tends to  $+\infty$   
and above which  $x_n$  tends to  $-\infty$ .

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

To look for a solution formula, the most basic first step is :  
write out enough terms at the beginning of the solution sequence  
until you see a pattern developing.

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_1 = 4x_0 - 0.6d = 4a - 0.6d$$

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$$x_2 = 4x_1 - 0.6d = 4^2 a - 4 \times 0.6d - 0.6d$$

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$$x_3 = 4x_2 - 0.6d = 4^3 a - 4^2 \times 0.6d - 4 \times 0.6d - 0.6d$$

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$$x_3 = 4x_2 - 0.6d = 4^3 a - 4^2 \times 0.6d - 4 \times 0.6d - 0.6d$$

$$x_4 = 4x_3 - 0.6d = 4^4 a - 4^3 \times 0.6d - 4^2 \times 0.6d - 4 \times 0.6d - 0.6d$$

See a pattern ?

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

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See a pattern ?

A power of 4  
building up.



## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

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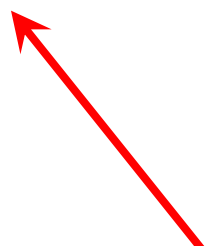
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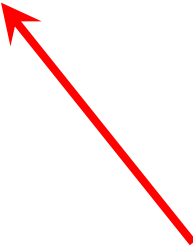
$$x_3 = 4x_2 - 0.6d = 4^3 a - (4^2 \times 0.6d + 4 \times 0.6d + 0.6d)$$

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$$x_n = 4^n a - (4^{n-1} + \cdots + 4^2 + 4 + 1)0.6d$$

## Example

Return to the model for an antibiotic treatment  
of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = 4^n a - (1 + 4 + 4^2 + \dots + 4^{n-1}) 0.6d$$



Can we simplify this sum ?

(so we don't have to add up so many terms to calculate  $x_n$  )

# Finite Geometric Series

(a short digression)

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$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

- A sum of  $n$  terms

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(a short digression)

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

- A sum of  $n$  terms
- Each term is  $r$  times its predecessor

Or equivalently ...

the ratio of a term to its predecessor is the  
*same* number  $r$  for all terms

## Finite Geometric Series

(a short digression)

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

- A sum of  $n$  terms
- Each term is  $r$  times its predecessor
- Initial term  $a$

## Finite Geometric Series

(a short digression)

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

There is a simpler formula available for calculating this sum.

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

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Subtract ...  $s_n - rs_n = a - ar^n$

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$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtract ...

$$s_n - rs_n = a - ar^n$$

Solve for ...

$$s_n = a \frac{1 - r^n}{1 - r}$$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = 4^n a - (1 + 4 + 4^2 + \cdots + 4^{n-1}) 0.6d$$

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Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

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A finite geometric series of  $n$  terms  
with ratio  $r = 4$  and first term  $a = 1$  .

## Example

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$$s_n = a \frac{1 - r^n}{1 - r} = \frac{1 - 4^n}{1 - 4}$$

## Example

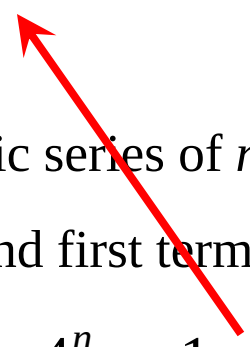
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A finite geometric series of  $n$  terms  
with ratio  $r = 4$  and first term  $a = 1$ .

$$s_n = a \frac{1 - r^n}{1 - r} = \frac{1 - 4^n}{1 - 4} = \frac{1}{3}(4^n - 1)$$


## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

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## Example

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$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = 4^n a - (4^n - 1)0.2d \quad \text{a solution formula}$$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

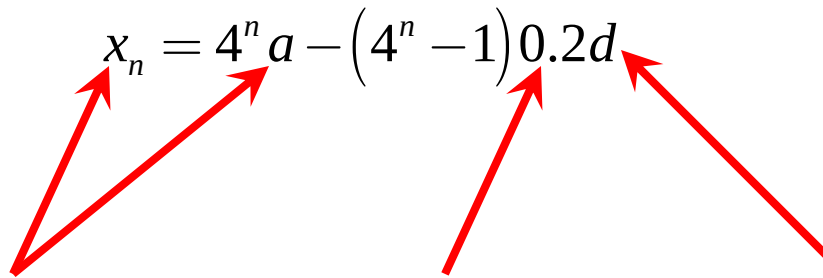
$$x_0 = a$$

$$x_n = 4^n a - (4^n - 1)0.2d$$

number  
of bacteria

number per  
milligram

milligrams



## Example

Return to the model for an antibiotic treatment  
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number  
of bacteria

number  
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Units balance  
on both sides  
of the solution  
formula

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Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

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Recall: we are interested in what happens to  $x_n$  as  $n \rightarrow +\infty$ .

So let's package the two  $4^n$  terms together :

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = (a - 0.2d)4^n + 0.2d$$

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Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

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$a - 0.2d > 0$  implies  $x_n \rightarrow +\infty$  as  $n \rightarrow +\infty$

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$a - 0.2d > 0$  implies  $x_n \rightarrow +\infty$  as  $n \rightarrow +\infty$

$a - 0.2d < 0$  implies  $x_n \rightarrow -\infty$  as  $n \rightarrow +\infty$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = (a - 0.2d)4^n + 0.2d$$

$$d < \frac{a}{0.2} \text{ implies } x_n \rightarrow +\infty \text{ as } n \rightarrow +\infty$$

$$d > \frac{a}{0.2} \text{ implies } x_n \rightarrow -\infty \text{ as } n \rightarrow +\infty$$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

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threshold for  $d$

$$\frac{\text{number}}{\text{number/mg}} = \text{mg}$$

## Example

Return to the model for an antibiotic treatment of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

$$x_n = (a - 0.2d)4^n + 0.2d$$

$d < 5a$  implies  $x_n \rightarrow +\infty$  as  $n \rightarrow +\infty$

$d > 5a$  implies  $x_n \rightarrow -\infty$  as  $n \rightarrow +\infty$

This shows the conjecture is true ...  
and that the threshold dosage is  $5a$  mg

## Example

Return to the model for an antibiotic treatment  
of an infection by *Staphylococcus aureus*.

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

The threshold dosage is  $5a$  mg.

This corroborates the two examples that led to the conjecture :

$$a = 1 \text{ and } d = 1$$

threshold = 5

treatment fails

$$a = 1 \text{ and } d = 5.1$$

threshold = 5

treatment succeeds

In this example we encountered *finite* geometric series.

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

These series arise in many applications.

Here's another ...

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Return to the antibiotic treatment problem.

In the previous example, we assumed no carry over of antibiotic from one treatment period to the next.

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Suppose instead that the antibiotic can stay in effect for longer than the time between treatments

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Return to the antibiotic treatment problem.

In the previous example, we assumed no carry over of antibiotic from one treatment period to the next.

Suppose instead that the antibiotic can stay in effect for longer than the time between treatments

... so there is an accumulation of antibiotic in the patient over time.

We want to determine how much antibiotic is present at any time, (in order to make sure that, in the long run, it lies between acceptable limits).

## Example

Assume 95% of the antibiotic  
is used or otherwise eliminated between treatments.

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$y_n$  = amount (mg) antibiotic in patient at time  $n = 0, 1, 2, 3, \dots$

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$$y_{n+1} = 0.05y_n + d$$

Question:

What dosages  $d$  will in the long run keep  $y_n$  between 250 - 300 mg?

## Example

$$y_{n+1} = 0.05y_n + d$$

$y_0 =$  initial amount in patient we assume  $= 0$

## Example

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$$y_2 = 0.05y_1 + d$$

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$$y_3 = 0.05y_2 + d$$

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$$y_4 = 0.05y_3 + d$$

### Example

$$y_{n+1} = 0.05y_n + d$$

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observe pattern

### Example

$$y_{n+1} = 0.05y_n + d$$

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$\vdots$

$$y_n = 0.05^{n-1}d + 0.05^{n-2}d + \cdots + 0.05d + d$$

### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

the solution formula  
is a finite geometric series

### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

What happens in the long run?

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} \left( d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d \right) = ?$$

### Example

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$$\lim_{n \rightarrow +\infty} (d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d)$$

This is called an *infinite series*.

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$$y_{n+1} = 0.05y_n + d$$

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This is called an *infinite series*.

$$s_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

The *sequence of partial sums*.

## Definition

An infinite series  $\lim_{n \rightarrow +\infty} (a_0 + a_1 + a_2 + \cdots + a_{n-1})$  *converges* if (and only if) its sequence of partial sums  $s_n = a_0 + a_1 + a_2 + \cdots + a_{n-1}$  converges.

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## Some notation

$$\lim_{n \rightarrow +\infty} (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$


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## Some notation

$$\lim_{n \rightarrow +\infty} (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = a_0 + a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=0}^{\infty} a_n$$



“sigma notation”

## Definition

An infinite series  $\sum_{i=0}^{\infty} a_i$  *converges* if (and only if)

its sequence of partial sums  $s_n = \sum_{i=0}^{n-1} a_i$  converges.

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An infinite series  $\sum_{i=0}^{\infty} a_i$  *diverges* if (and only if)

its sequence of partial sums  $s_n = \sum_{i=0}^{n-1} a_i$  diverges.

### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

What happens in the long run?

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} \left( d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d \right) = ?$$

### Example

$$y_{n+1} = 0.05y_n + d$$

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What happens in the long run?

$$\lim_{n \rightarrow +\infty} y_n = \sum_{i=0}^{\infty} 0.05^i d = ?$$



- Does this infinite series converge ?
- If so, to what ?

### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

What happens in the long run?

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By definition: 
$$\sum_{i=0}^{\infty} 0.05^i d = \lim_{n \rightarrow +\infty} \sum_{i=0}^{n-1} 0.05^i d$$

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A finite geometric series

### Example

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By definition:  $\sum_{i=0}^{\infty} 0.05^i d = \lim_{n \rightarrow +\infty} \sum_{i=0}^{n-1} 0.05^i d$

Recall:

$$\sum_{i=0}^{n-1} ar^i = a \frac{1-r^n}{1-r}$$

A finite geometric series



### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

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### Example

$$y_{n+1} = 0.05y_n + d$$

$$y_n = d + 0.05d + 0.05^2d + \cdots + 0.05^{n-1}d$$

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What dosages  $d$  will in the long run keep  $y_n$  between 250 - 300 mg?

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Answer:

$$250 \leq \frac{19}{20}d \leq 300$$

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Answer:

$$\frac{5000}{19} \leq d \leq \frac{6000}{19}$$

### Example

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Answer:

$$263.2 \text{ mg} \approx \frac{5000}{19} \leq d \leq \frac{6000}{19} \approx 315.8 \text{ mg}$$

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For geometric series, we know the answer to both:

$$\sum_{i=0}^{\infty} ar^i \text{ converges to } \frac{a}{1-r} \text{ if } |r| < 1$$

$$\sum_{i=0}^{\infty} ar^i \text{ diverges if } |r| \geq 1$$

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Second question is in general much more difficult.

Some relevant methods of calculation are studied in Chapter 10.