

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, October 25

<http://math.arizona.edu/~cushing/250a.html>

Chapter 9.1

Introduction to Sequences & Discrete Dynamical Systems

Example

Cultures of the bacterium *Staphylococcus aureus* double every 30 minutes.

Let $x_n =$ number of bacteria at time n .

Measure time in hours: $n = 0, 1, 2, 3, \dots$.

Measure bacteria in units of one million 10^6 .

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$$x_{n+1} = 4x_n$$

Bacteria numbers quadruple every hour.

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$$x_0 = a$$

Initial infection by $a \times 10^6$ bacteria at hour $n = 0$.

$$x_{n+1} = 4x_n$$

$$x_0 = a$$

This is an example of an *initial value problem*.

Each *initial condition* a generates a unique sequence
(of predicted bacteria numbers),
called the *solution* of the initial value problem :

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$$\vdots$$

$$x_n = 4^n a$$

$$x_{n+1} = 4x_n$$

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This is an example of an *initial value problem*.

Each *initial condition* a generates a unique sequence
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called the *solution* of the initial value problem :

$$x_n = 4^n a$$

This is called a *solution formula*.

$$x_{n+1} = 4x_n$$

$$x_0 = a$$

$$x_n = 4^n a$$

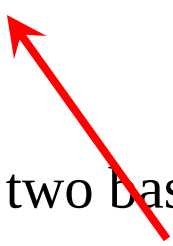
These illustrate two basic ways to define numerical sequences:

$$x_{n+1} = 4x_n$$

$$x_0 = a$$

$$x_n = 4^n a$$

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recursively



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These illustrate two basic ways to define numerical sequences:

recursively



$$x_{n+1} = f(x_n)$$

$$x_0 = a$$

$$x_{n+1} = 4x_n$$

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$$x_n = 4^n a$$

These illustrate two basic ways to define numerical sequences:
 recursively or by formulas



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$$x_{n+1} = f(x_n)$$

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$$x_{n+1} = g(n)$$

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What dosage d will cure the infection ?

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What dosage d will cure the infection ?

Since we measure x in millions, each hour the antibiotic treatment kills

$$\frac{6 \times 10^5 d}{10^6} = 0.6 d \text{ million bacteria}$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

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Take $a = 1$ and $d = 1$:

$$x_{n+1} = 4x_n - 0.6$$

$$x_0 = 1$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

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Take $a = 1$ and $d = 1$:

$$x_{n+1} = 4x_n - 0.6$$

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$$x_1 = 4x_0 - 0.6 = 3.4$$

Example

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Example

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Example

$$x_{n+1} = 4x_n - 0.6d$$

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$$x_4 = 4x_3 - 0.6 = 205.0$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

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$$x_4 = 4x_3 - 0.6 = 205.0$$

Does not appear to cure the infection !

Example

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

Increased
dosage

Take $a = 1$ and $d = 1$:

$$x_{n+1} = 4x_n - 0.6$$

$$x_0 = 1$$

$$x_1 = 4x_0 - 0.6 = 3.4$$

$$x_2 = 4x_1 - 0.6 = 13.0$$

$$x_3 = 4x_2 - 0.6 = 51.4$$

$$x_4 = 4x_3 - 0.6 = 205.0$$

Take $a = 1$ and $d = 5.1$:

$$x_{n+1} = 4x_n - 3.06$$

$$x_0 = 1$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

Take $a = 1$ and $d = 1$:

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Take $a = 1$ and $d = 5.1$:

$$x_{n+1} = 4x_n - 3.06$$

$$x_0 = 1$$

$$x_1 = 4x_0 - 3.06 = 0.94$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

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Take $a = 1$ and $d = 1$:

$$x_{n+1} = 4x_n - 0.6$$

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Take $a = 1$ and $d = 5.1$:

$$x_{n+1} = 4x_n - 3.06$$

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$$x_2 = 4x_1 - 3.06 = 0.7$$

Example

$$x_{n+1} = 4x_n - 0.6d$$

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Take $a = 1$ and $d = 5.1$:

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Example

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Take $a = 1$ and $d = 5.1$:

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We interpret this to mean that
bacteria numbers dropped to 0
between hour 2 and 3.

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$$x_{n+1} = 4x_n - 0.6d$$

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Take $a = 1$ and $d = 1$:

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Take $a = 1$ and $d = 5.1$:

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Conjecture:

There is a threshold for the dosage d
(somewhere between $d = 1$ and $d = 5.1$)
below which the treatment does *not* cure the infection
and above which it does .

Example

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Conjecture:

There is a threshold for the dosage d
below which the treatment does *not* cure the infection is
and above which it does .

The threshold depends on the initial infection number a

Example

$$x_{n+1} = 4x_n - 0.6d$$

$$x_0 = a$$

Mathematically the conjecture is :

There is a threshold value of d
below which the solution x_n tends to $+\infty$
and above which x_n tends to $-\infty$.

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Mathematically the conjecture is :

There is a threshold value of d

below which the solution x_n tends to $+\infty$

and above which x_n tends to $-\infty$.

We could (probably easily) prove this conjecture true or false,

(and, if true, determine the threshold dosage d)

if we had a *solution formula*

$$x_n = g(n).$$

Example

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Turns out it's true that either the solution x_n either tends to $+\infty$ or to $-\infty$ as $n \rightarrow \infty$.

In other words, solution sequences *diverge*.

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$$x_{n+1} = r x_n, \quad r > 0$$

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Solution formula: $x_n = r^n x_0$

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We say the sequence *diverges* .

Example

$$x_{n+1} = r x_n, \quad r > 0$$

Solution formula: $x_n = r^n x_0$

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We say the sequence *diverges* .

We say the sequence *converges* .

Formal Definition

A sequence x_n *converges* if there is a number L
(called the *limit* of the sequence) for which the following is true :
for *any* small, positive number ε there is a number N so that

$$|x_n - L| < \varepsilon \text{ for all } n > N.$$

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The sequence gets and remains as close to L as you want,
provided you go far enough out the sequence.