

MATH 250a

Fall Semester 2007

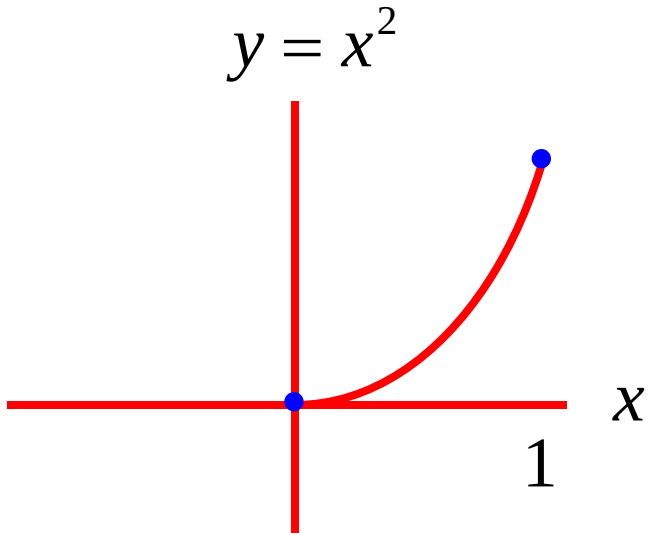
Section 2 (J. M. Cushing)

Thursday, October 18

<http://math.arizona.edu/~cushing/250a.html>

Lengths of Curves

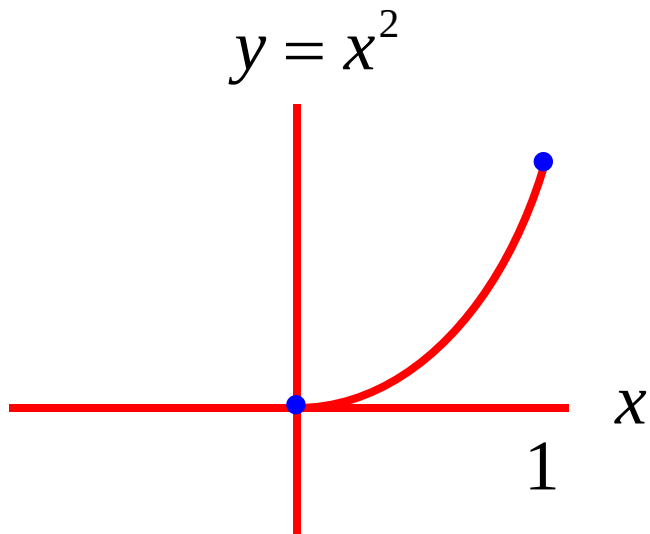
What's the length of the curve ?



Lengths of Curves

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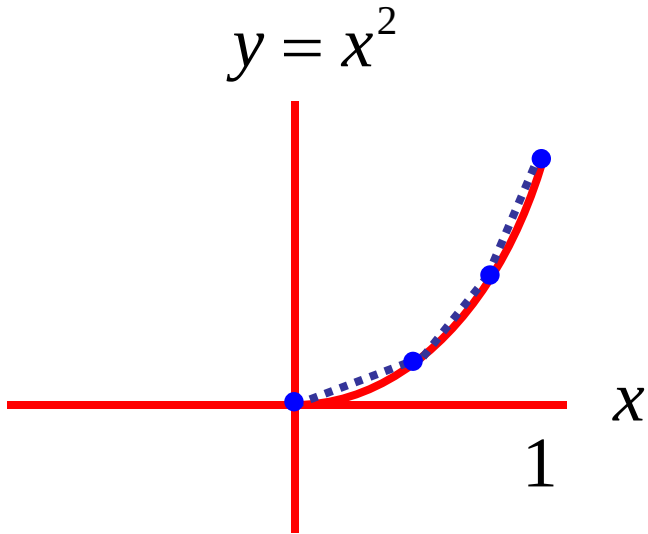
Approximate length by curve
of broken line segments



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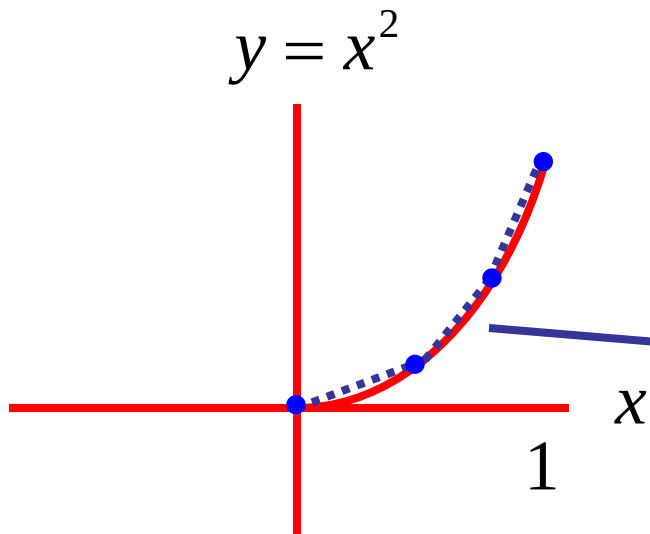
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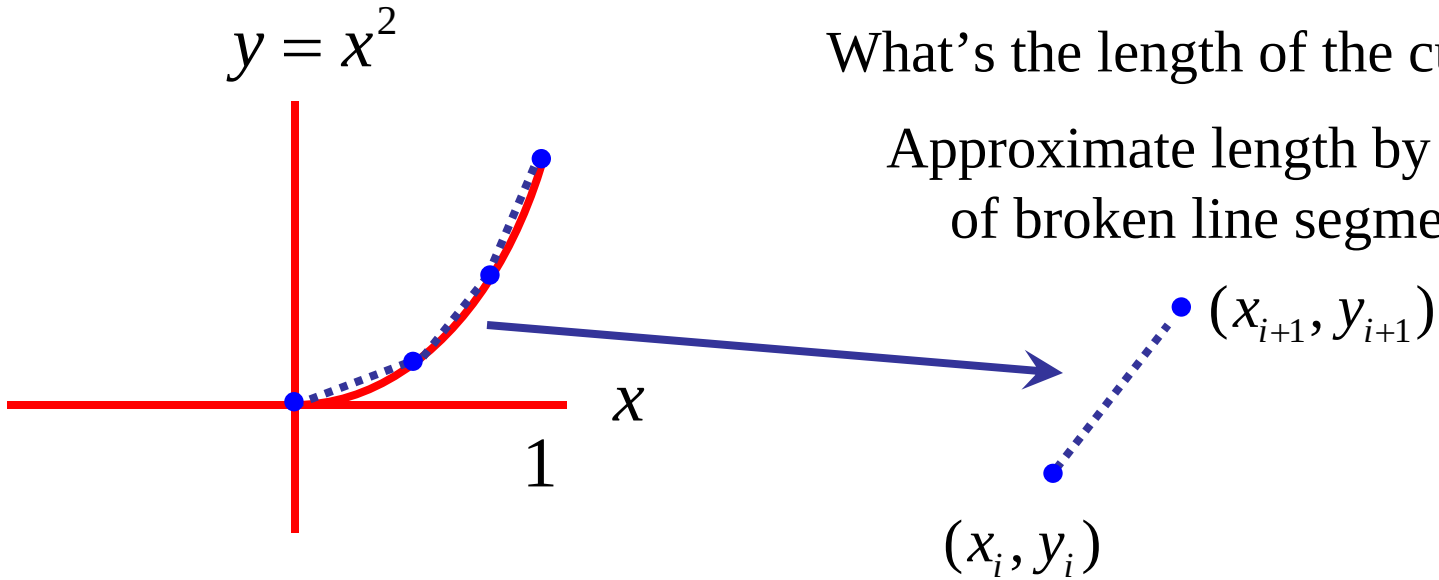
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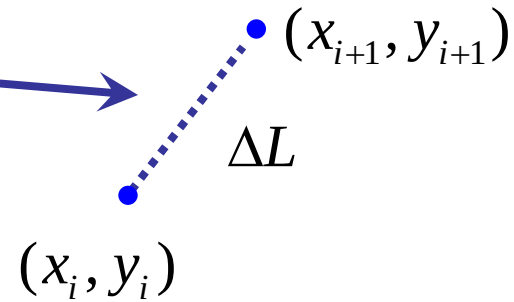
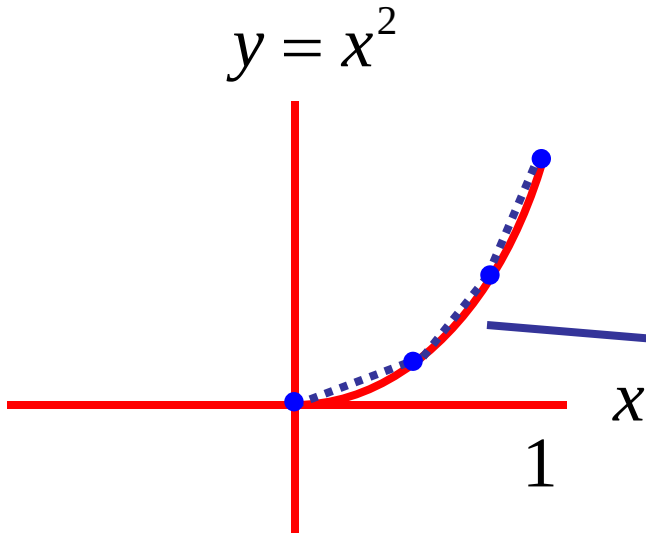


$$y_i = f(x_i) \text{ where } f(x) = x^2$$

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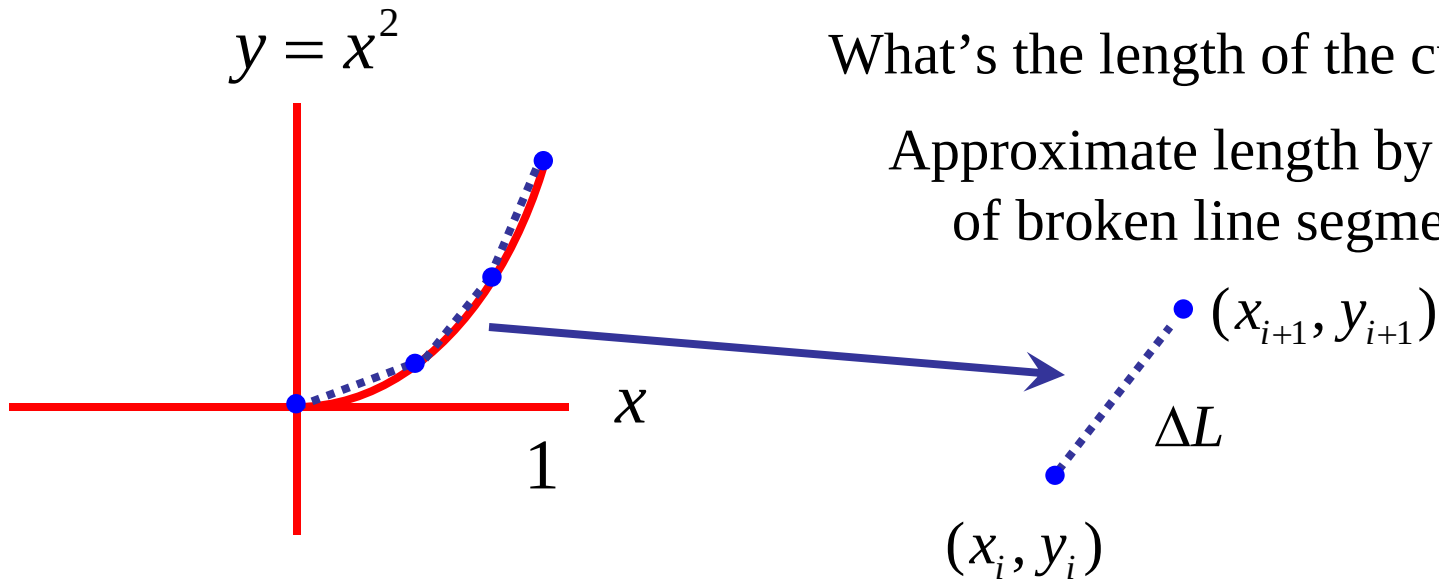
$$\Delta L = \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$$

Pythagoras' Theorem

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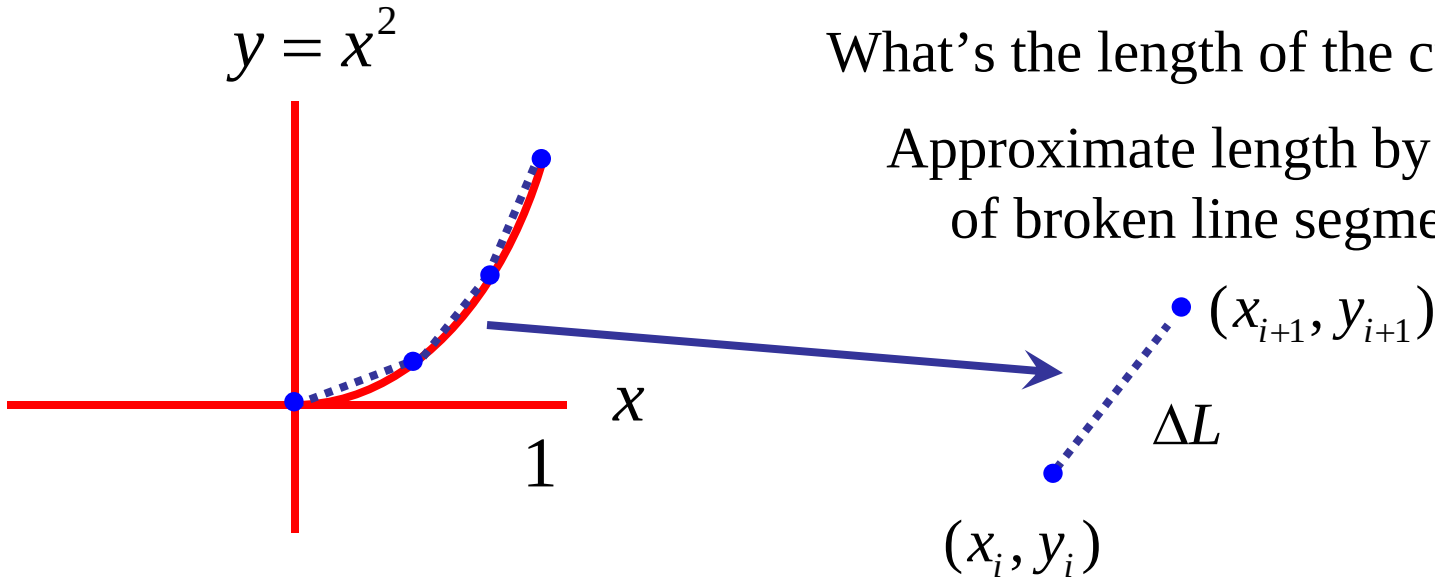
$$\Delta L = \sqrt{(x_{i+1} - x_i)^2 + (f(x_{i+1}) - f(x_i))^2}$$

Assume equally spaced x_i , so that $x_{i+1} - x_i = \text{constant} = \Delta x$

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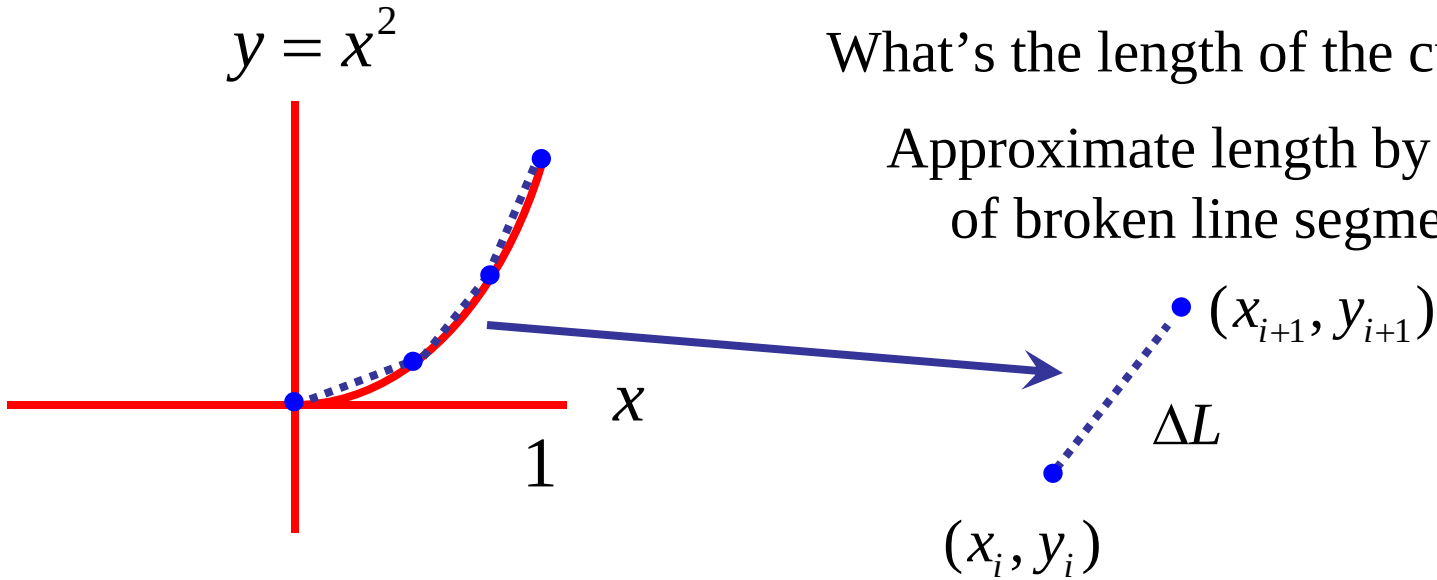
$$\Delta L = \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2}$$

Assume equally spaced x_i , so that $x_{i+1} - x_i = \text{constant} = \Delta x$

Lengths of Curves

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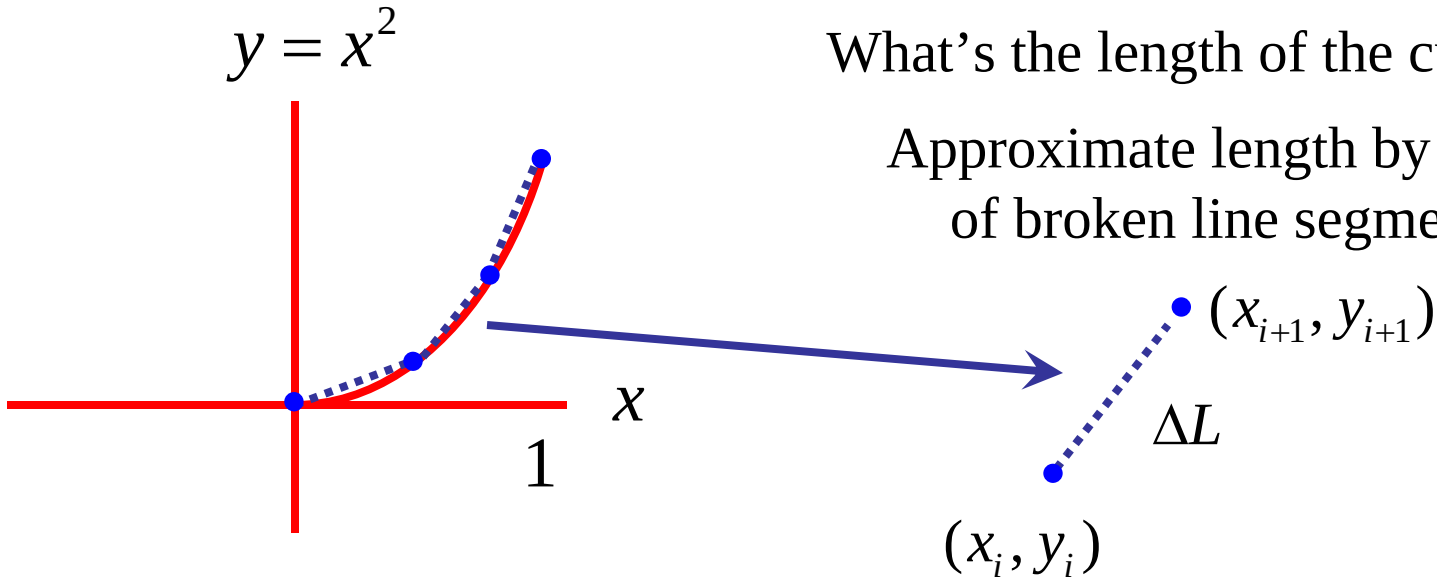
$$y_i = f(x_i) \text{ where } f(x) = x^2$$

$$\Delta L = \sqrt{(\Delta x)^2 + (f(x_i + \Delta x) - f(x_i))^2} = \sqrt{1 + \left(\frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \right)^2} \Delta x$$

Lengths of Curves

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Approximate length by curve
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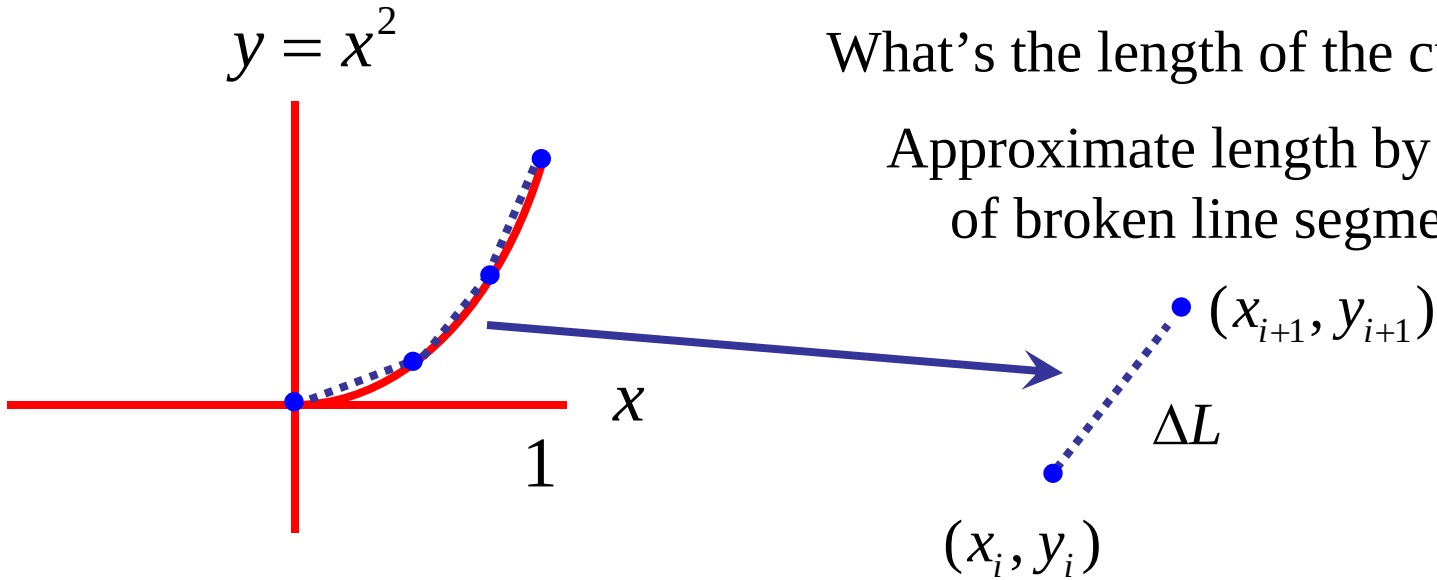
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Δx
 $\nearrow$
 $\approx f'(x_i)$

Lengths of Curves

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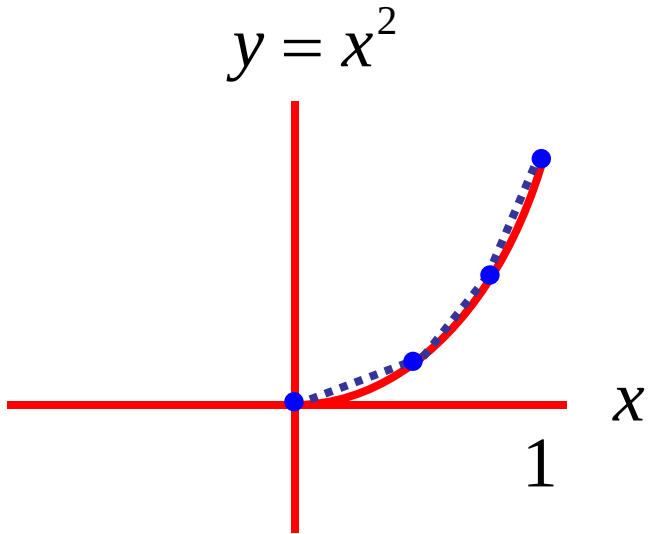


$$y_i = f(x_i) \text{ where } f(x) = x^2$$

$$\Delta L \approx \sqrt{1 + (f'(x_i))^2} \Delta x$$

Lengths of Curves

What's the length of the curve ?



$$L \approx \sum \Delta L \approx \sum \sqrt{1 + (f'(x_i))^2} \Delta x$$

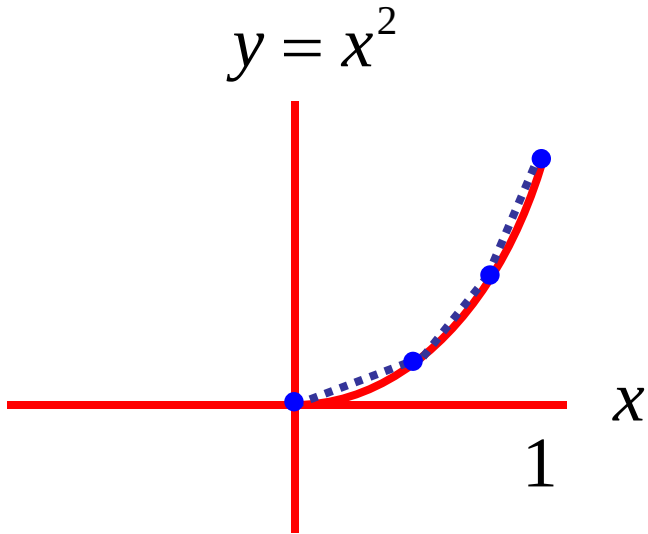
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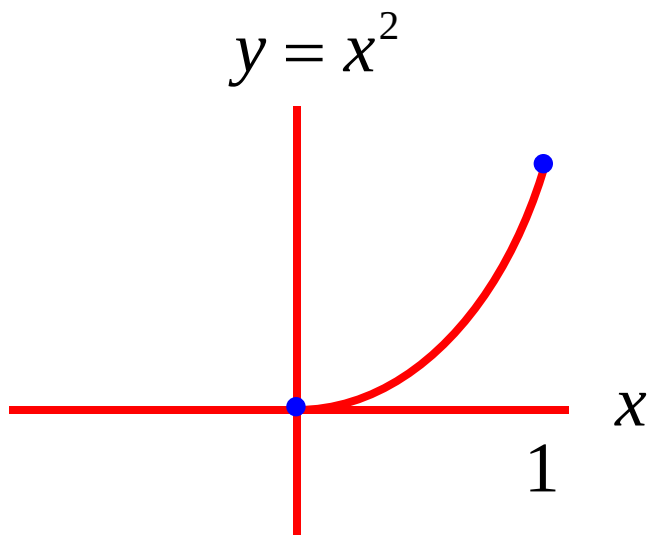
where $f(x) = x^2$



Pass the number of line segments to infinity

Lengths of Curves

What's the length of the curve ?

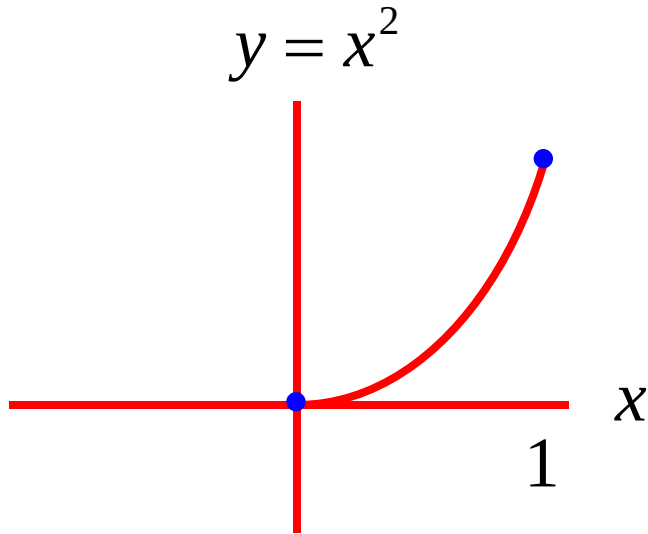


$$L = \int_0^1 \sqrt{1 + (f'(x))^2} \, dx$$

where $f(x) = x^2$

Lengths of Curves

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$$L = \int_0^1 \sqrt{1 + (f'(x))^2} \, dx$$

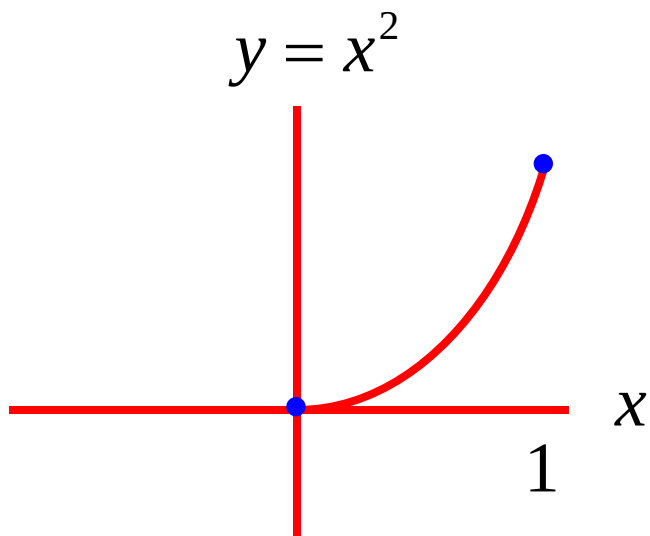
$$\text{where } f(x) = x^2$$

In general, the length of a curve $y = f(x)$
between points $(a, f(a))$ and $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

Lengths of Curves

What's the length of the curve ?

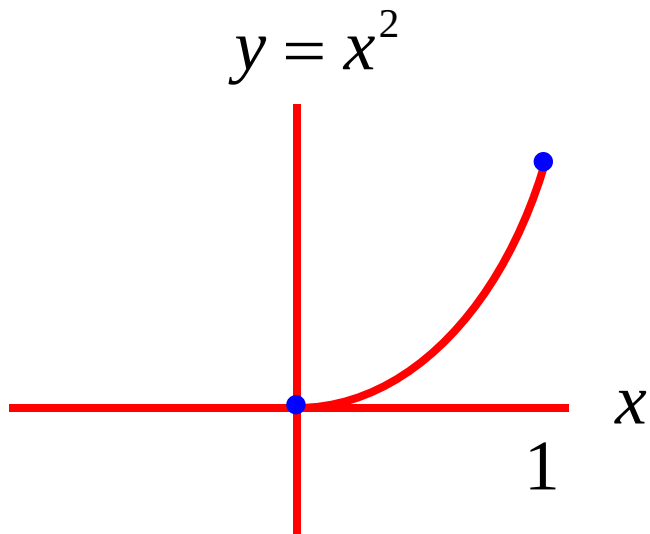


$$L = \int_0^1 \sqrt{1 + 4x^2} \, dx$$

where $f(x) = x^2$

Lengths of Curves

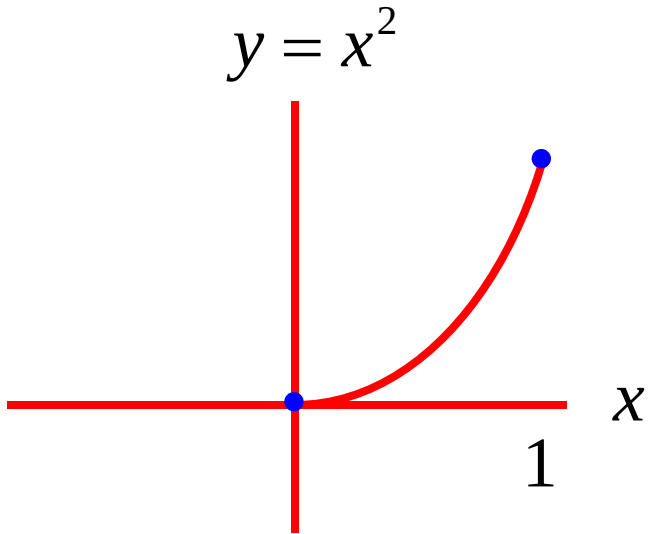
What's the length of the curve ?



$$L = \int_0^1 \sqrt{1 + 4x^2} \, dx = 2 \int_0^1 \sqrt{\frac{1}{4} + x^2} \, dx$$

Lengths of Curves

What's the length of the curve ?



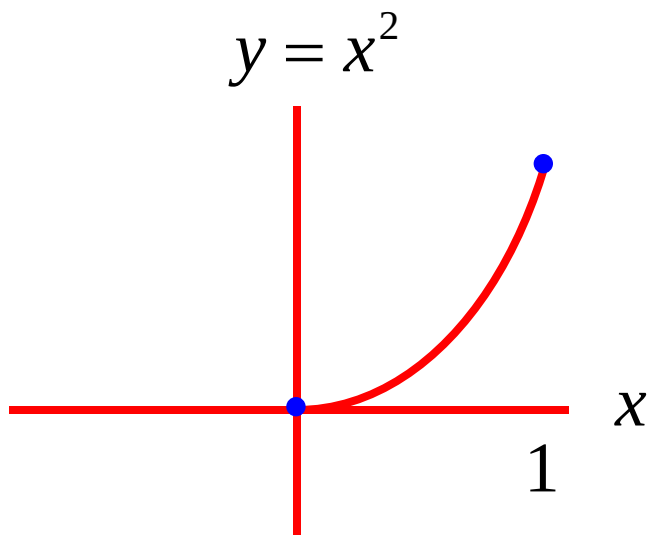
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Table VI, #30 and #29.

$$\int \sqrt{\frac{1}{4} + x^2} \, dx = \frac{1}{2} x \sqrt{\frac{1}{4} + x^2} + \frac{1}{8} \ln \left(x + \sqrt{\frac{1}{4} + x^2} \right)$$

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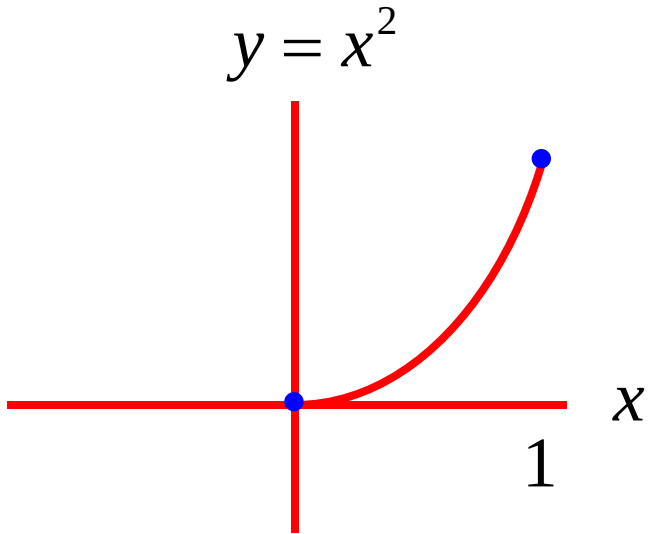
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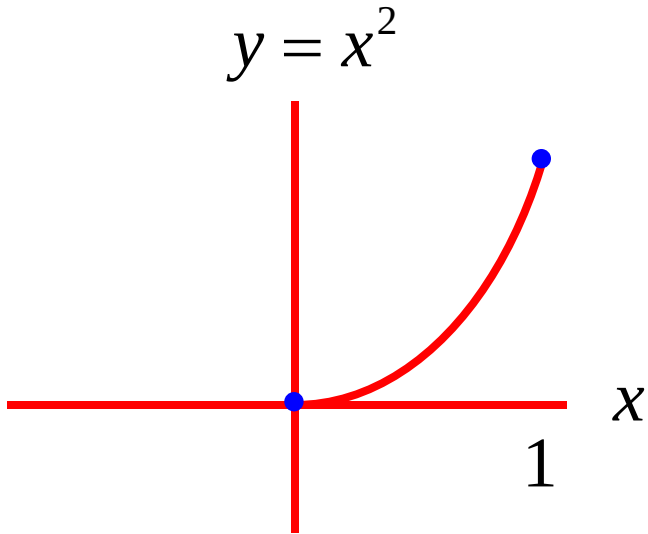
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Lengths of Curves

What's the length of the curve ?



$$L = \int_0^1 \sqrt{1 + 4x^2} \, dx = 2 \int_0^1 \sqrt{\frac{1}{4} + x^2} \, dx$$

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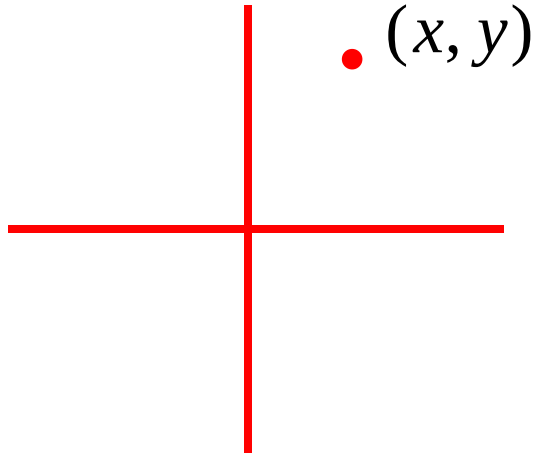
$$\begin{aligned} L &= \frac{1}{2} x \sqrt{\frac{1}{4} + x^2} + \frac{1}{8} \ln \left(x + \sqrt{\frac{1}{4} + x^2} \right) \Big|_0^1 \\ &= \frac{1}{8} \ln 2 + \frac{1}{4} \sqrt{5} + \frac{1}{8} \ln \left(\frac{\sqrt{5}}{2} + 1 \right) \approx 0.7395 \end{aligned}$$

Lengths & Areas in Polar Coordinates

Some curves and areas are better described in polar coordinates r and θ rather than rectangular coordinates x and y .

Lengths & Areas in Polar Coordinates

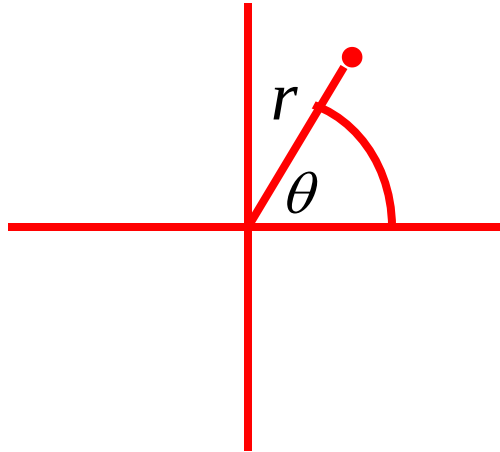
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Location of a point (not at the origin) is determined by its rectangular coordinates

Lengths & Areas in Polar Coordinates

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Location of a point (not at the origin) is determined by its rectangular coordinates

... or, equally as well, by its polar coordinates

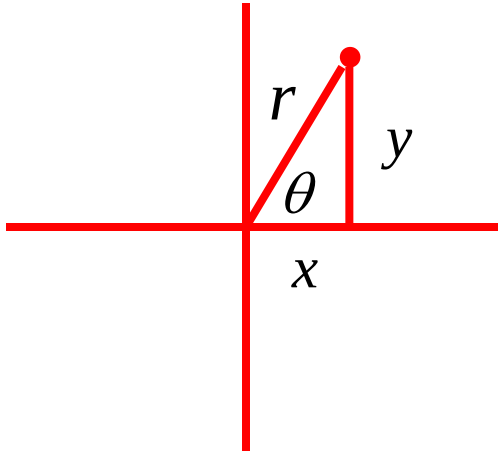
Lengths & Areas in Polar Coordinates

Some curves and areas are better described in polar coordinates r and θ rather than rectangular coordinates x and y .

Relationship is from basic trigonometry :

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$-\pi < \theta \leq \pi$$



Lengths & Areas in Polar Coordinates

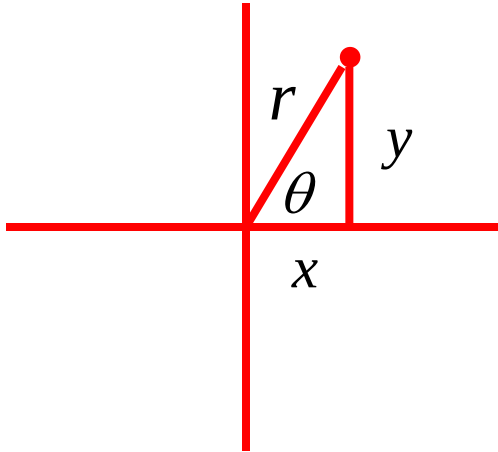
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These permit a change to *polar* coordinates in mathematical expressions expressed in *rectangular* coordinates.



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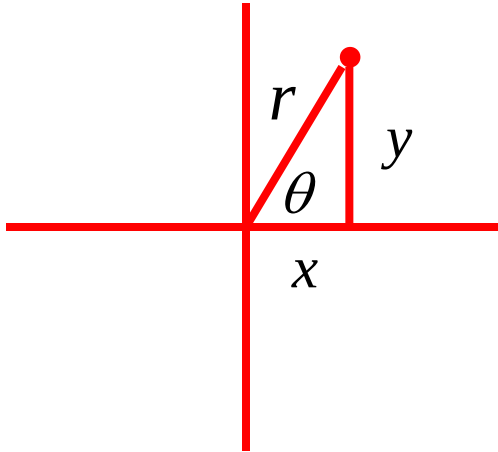
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Examples

A circle in rectangular coordinates: $x^2 + y^2 = 4$



Lengths & Areas in Polar Coordinates

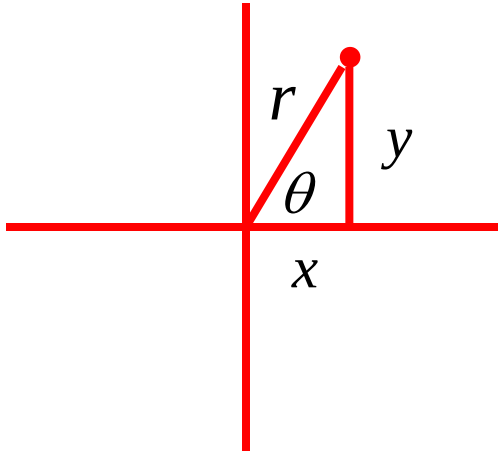
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in polar coordinates is: $(r \cos \theta)^2 + (r \sin \theta)^2 = 4$

Lengths & Areas in Polar Coordinates

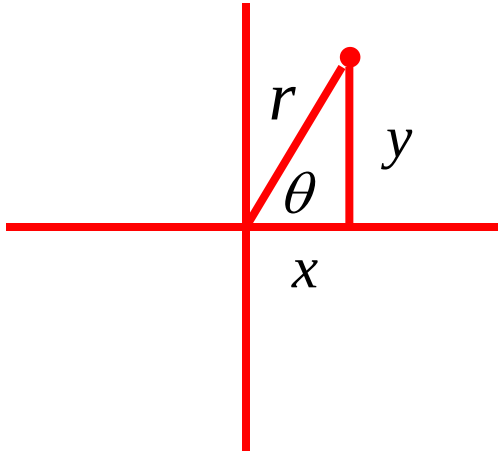
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Examples



A circle in rectangular coordinates: $x^2 + y^2 = 4$
in polar coordinates is: $(r \cos \theta)^2 + (r \sin \theta)^2 = 4$
or simply: $r = 2$

Lengths & Areas in Polar Coordinates

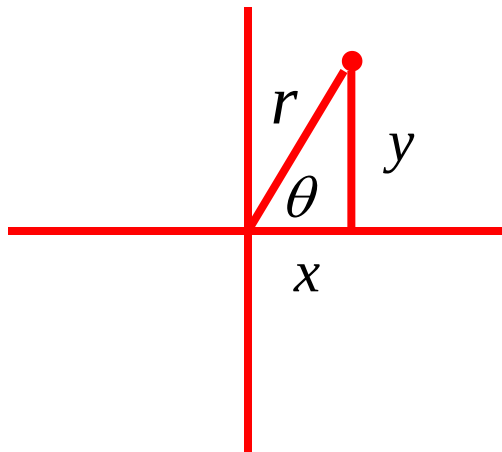
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Polar coordinates give “simpler” description.

Lengths & Areas in Polar Coordinates

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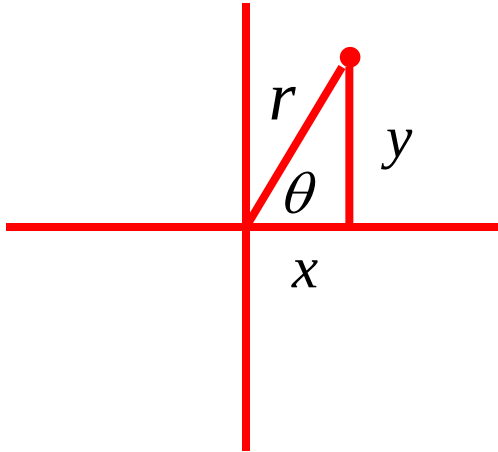
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Examples

The parabola in rectangular coordinates: $y = x^2$



Lengths & Areas in Polar Coordinates

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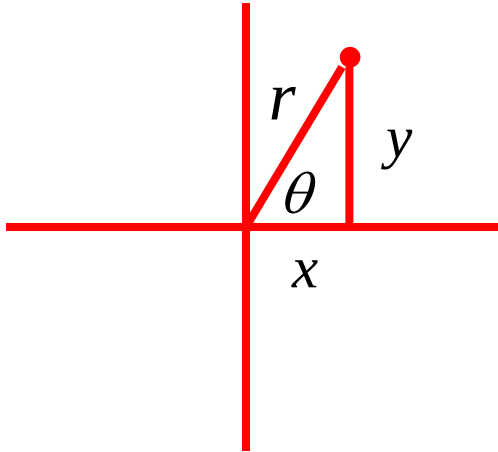
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The parabola in rectangular coordinates: $y = x^2$
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Lengths & Areas in Polar Coordinates

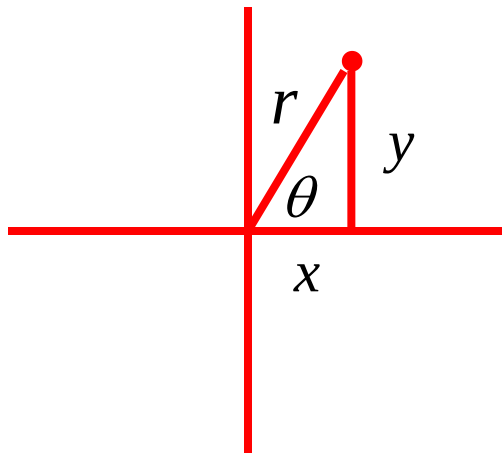
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Examples



The parabola in rectangular coordinates: $y = x^2$

in polar coordinates is: $r \sin \theta = (r \cos \theta)^2$

$$\text{or } r = 0 \quad \text{and} \quad r = \frac{\sin \theta}{\cos^2 \theta}$$

Lengths & Areas in Polar Coordinates

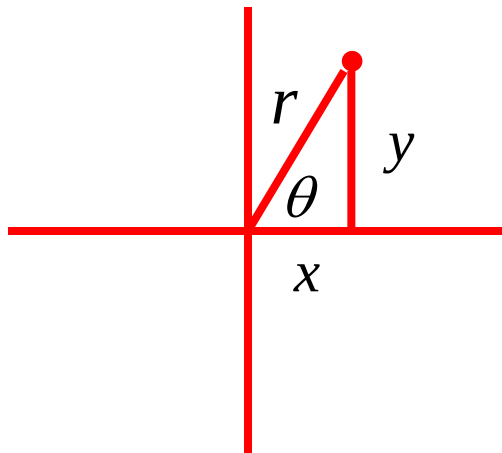
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Examples



The parabola in rectangular coordinates: $y = x^2$

in polar coordinates is: $r \sin \theta = (r \cos \theta)^2$

or $r = 0$ and $r = \frac{\sin \theta}{\cos^2 \theta}$

Rectangular coordinates give “simpler” description.

Lengths & Areas in Polar Coordinates

Some curves and areas are better described in polar coordinates r and θ rather than rectangular coordinates x and y .

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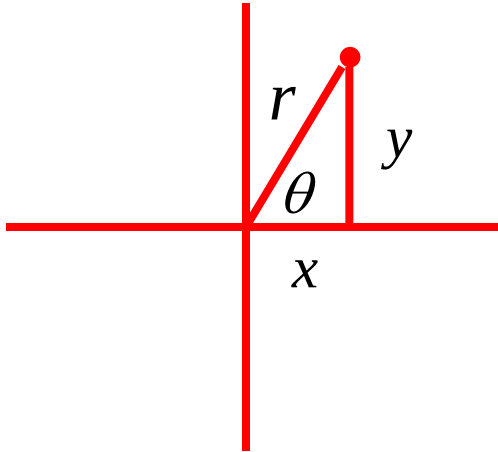
$$x = r \cos \theta, \quad y = r \sin \theta$$

$$-\pi < \theta \leq \pi$$

Convert rectangular to polar coordinates.

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}$$

Convert polar to rectangular coordinates.



Lengths & Areas in Polar Coordinates

$$r = 4 \quad \text{and} \quad r = \frac{\sin \theta}{\cos^2 \theta}$$

are examples of curves described in polar coordinates
by equations of the form

$$r = f(\theta)$$

Lengths & Areas in Polar Coordinates

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The rectangular coordinates are then

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta$$

Lengths & Areas in Polar Coordinates

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are examples of curves described in polar coordinates
by equations of the form

$$r = f(\theta)$$

The rectangular coordinates are then

$$x = f(\theta) \cos \theta, \quad y = f(\theta) \sin \theta$$

This is an example of a parameteric representation of a curve

$$x = x(\theta), \quad y = y(\theta)$$

Lengths & Areas in Polar Coordinates

Example

$$r = k \theta, \quad 0 \leq \theta < +\infty$$

radial distance is proportional to polar angle

Lengths & Areas in Polar Coordinates

Example

$$r = k \theta, \quad 0 \leq \theta < +\infty$$

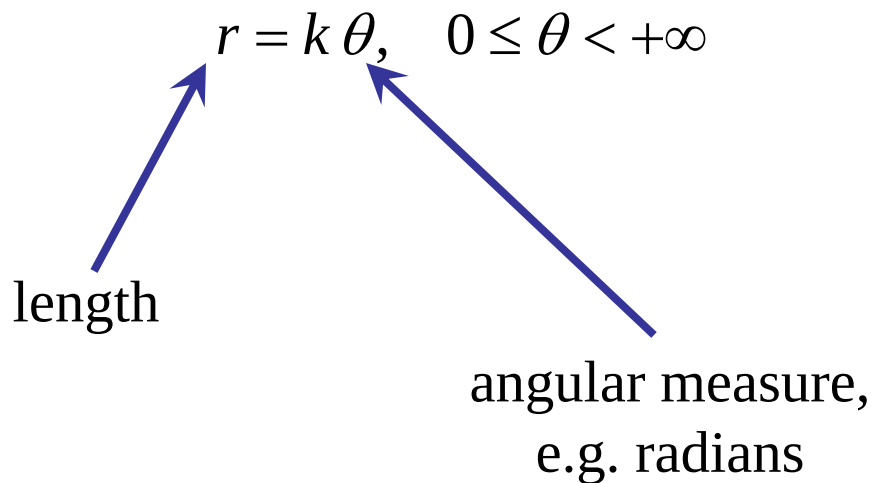
length 

Lengths & Areas in Polar Coordinates

Example

$$r = k \theta, \quad 0 \leq \theta < +\infty$$

length



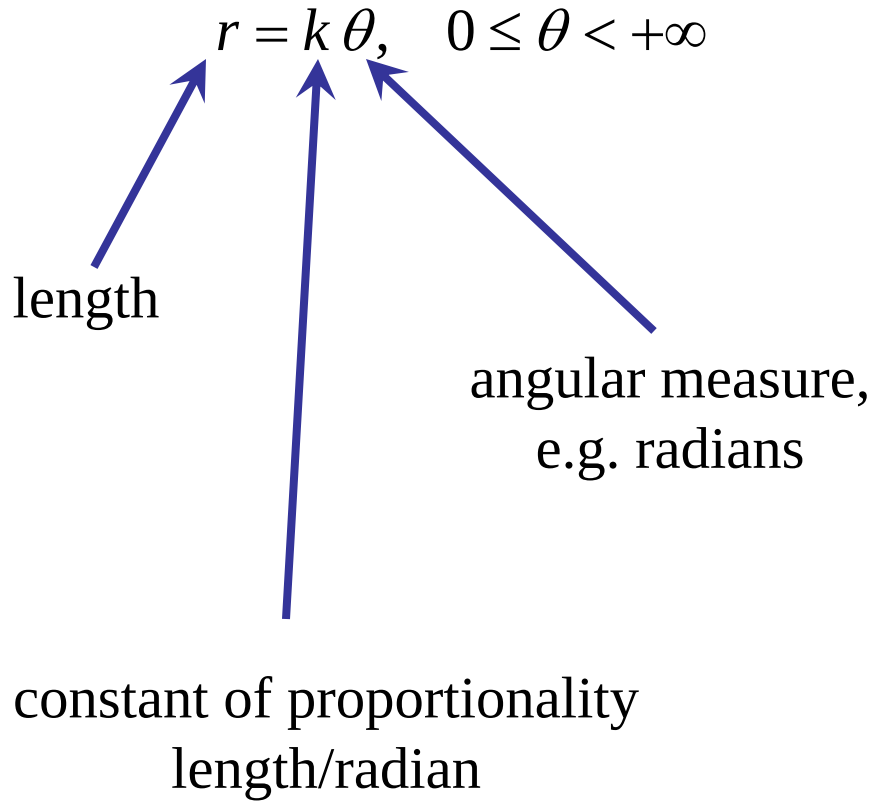
angular measure,
e.g. radians

Lengths & Areas in Polar Coordinates

Example

$$r = k \theta, \quad 0 \leq \theta < +\infty$$

length



angular measure,
e.g. radians

constant of proportionality
length/radian

Lengths & Areas in Polar Coordinates

Example

$$r = \theta, \quad 0 \leq \theta < +\infty$$

Lengths & Areas in Polar Coordinates

Example

$$r = \theta, \quad 0 \leq \theta < +\infty$$

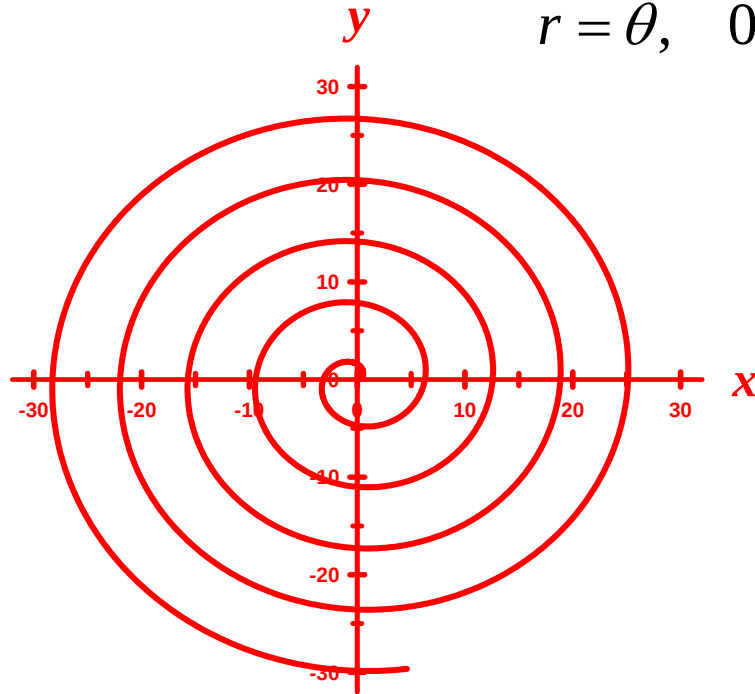


“invisible” coefficient 1 has units that account for a balance of unites on each side of equation

Lengths & Areas in Polar Coordinates

Example

$$r = \theta, \quad 0 \leq \theta < +\infty$$

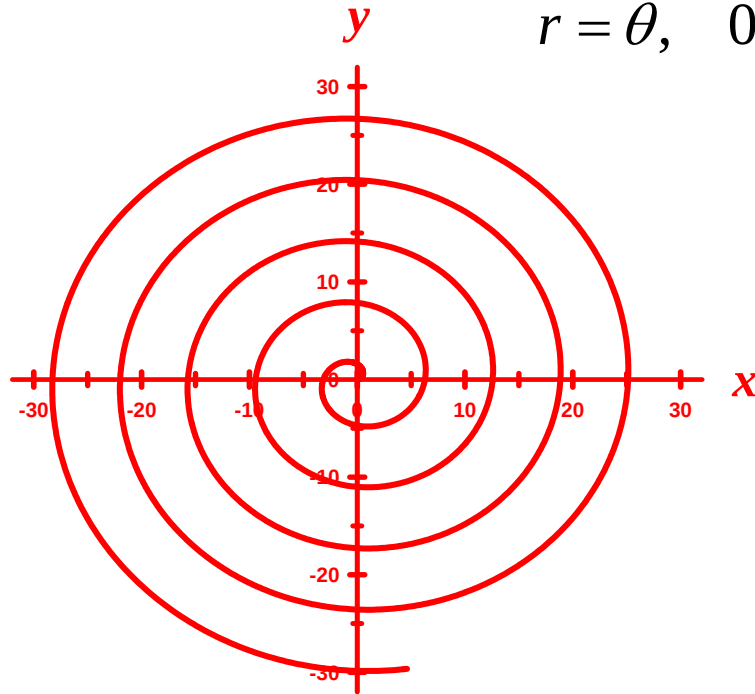


a spiral

Lengths & Areas in Polar Coordinates

Example

$$r = \theta, \quad 0 \leq \theta < +\infty$$



a spiral

Parameteric representation:

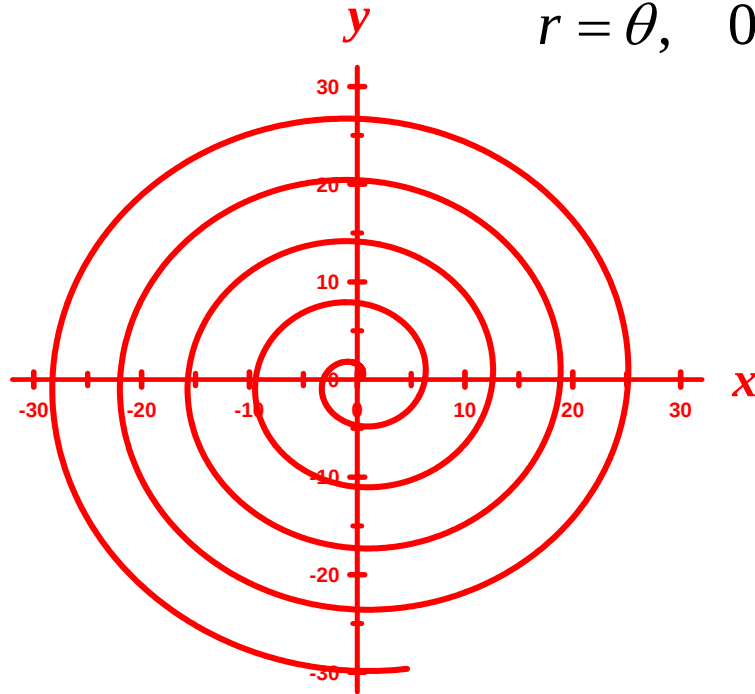
$$x = \theta \cos \theta, \quad y = \theta \sin \theta$$

$$0 \leq \theta < +\infty$$

Lengths & Areas in Polar Coordinates

Example

$$r = \theta, \quad 0 \leq \theta < +\infty$$



a spiral

Parameteric representation:

$$x = \theta \cos \theta, \quad y = \theta \sin \theta$$

$$0 \leq \theta < +\infty$$

Description in rectangular coordinates,

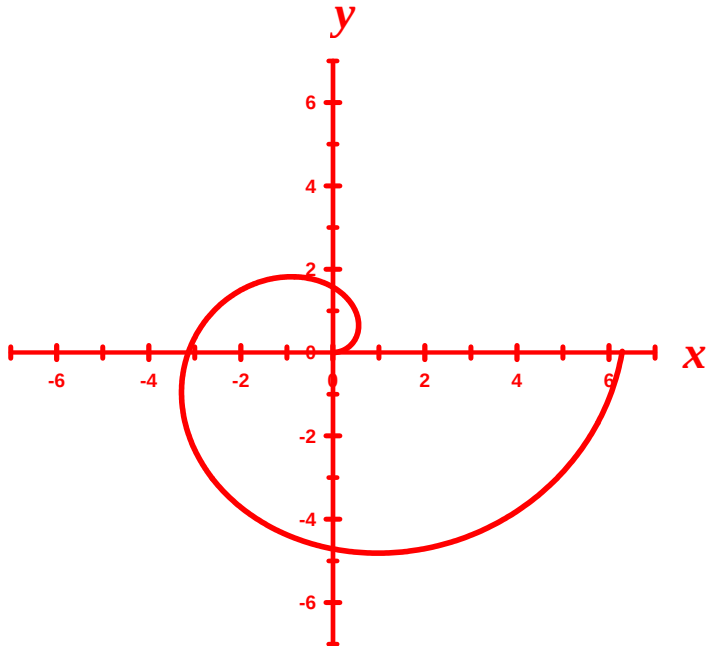
$$y = f(x)$$

while not impossible
is much more complicated.

Lengths & Areas in Polar Coordinates

Problem

What is the *length* of the section $r = \theta$, $0 \leq \theta \leq 2\pi$ of the spiral ?



Parametric representation:

$$x = \theta \cos \theta, \quad y = \theta \sin \theta$$

$$0 \leq \theta \leq 2\pi$$

Lengths & Areas in Polar Coordinates

A digression:

Formula for length of curve in rectangular coordinates $y = f(x)$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

Lengths & Areas in Polar Coordinates

A digression:

Formula for length of curve in rectangular coordinates $y = f(x)$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

What is the formula when given a parametric representation of the curve ?

$$x = x(\theta), \quad y = y(\theta)$$

$$\alpha \leq \theta \leq \beta$$

Lengths & Areas in Polar Coordinates

A digression:

Formula for length of curve in rectangular coordinates $y = f(x)$ is

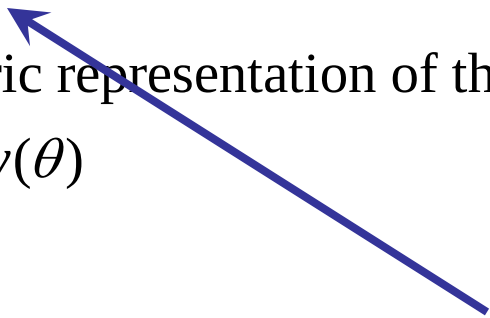
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Lengths & Areas in Polar Coordinates

A digression:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

By the Chain Rule: $\frac{dy}{d\theta} = \frac{dy}{dx} \frac{dx}{d\theta}$

Lengths & Areas in Polar Coordinates

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Lengths & Areas in Polar Coordinates

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Lengths & Areas in Polar Coordinates

A digression:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$$L = \int \sqrt{1 + \left(\frac{y'(\theta)}{x'(\theta)} \right)^2}$$

By the Chain Rule: $\frac{dy}{d\theta} = \frac{dy}{dx} \frac{dx}{d\theta}$

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Lengths & Areas in Polar Coordinates

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$$\frac{dx}{d\theta} = x'(\theta) \Rightarrow dx = x'(\theta) d\theta$$

Lengths & Areas in Polar Coordinates

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$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

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Lengths & Areas in Polar Coordinates

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θ limits !

Lengths & Areas in Polar Coordinates

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$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{y'(\theta)}{x'(\theta)} \right)^2} x'(\theta) d\theta$$

θ limits !

NOT x limits !

(a common student error !)

Lengths & Areas in Polar Coordinates

A digression:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{y'(\theta)}{x'(\theta)} \right)^2} x'(\theta) d\theta$$

Lengths & Areas in Polar Coordinates

A digression:

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left[1 + \frac{(y'(\theta))^2}{(x'(\theta))^2}\right] (x'(\theta))^2} d\theta$$

Lengths & Areas in Polar Coordinates

A digression:

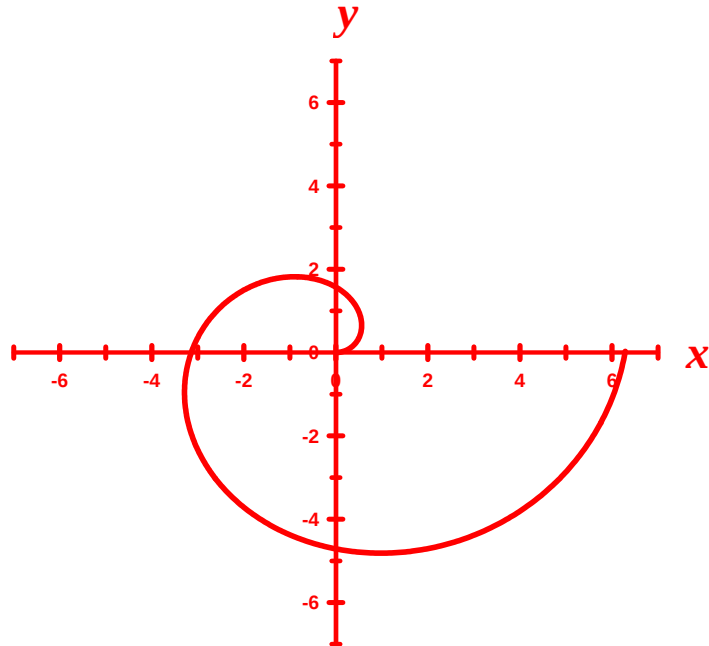
$$L = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(\theta))^2 + (y'(\theta))^2} \, d\theta$$

Lengths & Areas in Polar Coordinates

Problem

What is the length of the section $r = \theta$, $0 \leq \theta \leq 2\pi$ of the spiral ?



Parametric representation:

$$x = \theta \cos \theta, \quad y = \theta \sin \theta$$

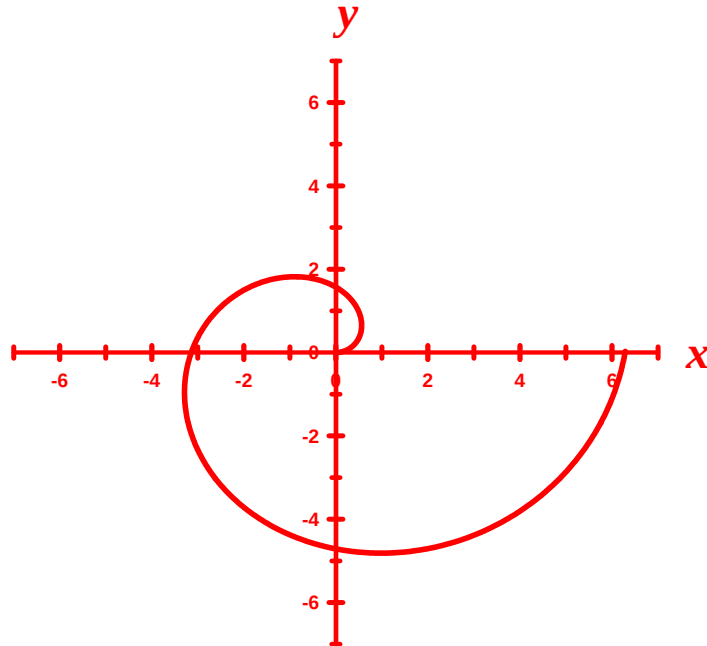
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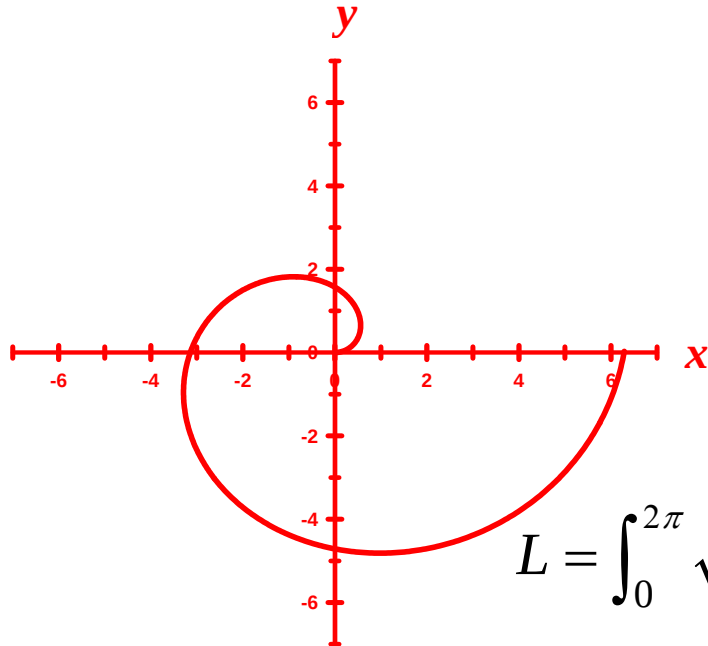
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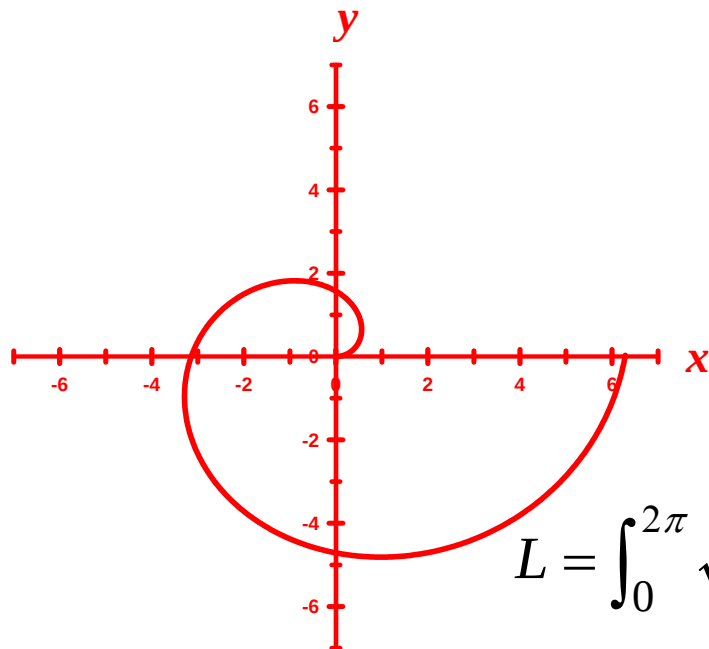
$$L = \int_{\alpha}^{\beta} \sqrt{(x'(\theta))^2 + (y'(\theta))^2} d\theta$$

$$L = \int_0^{2\pi} \sqrt{(-\theta \sin \theta + \cos \theta)^2 + (\theta \cos \theta + \sin \theta)^2} d\theta$$

Lengths & Areas in Polar Coordinates

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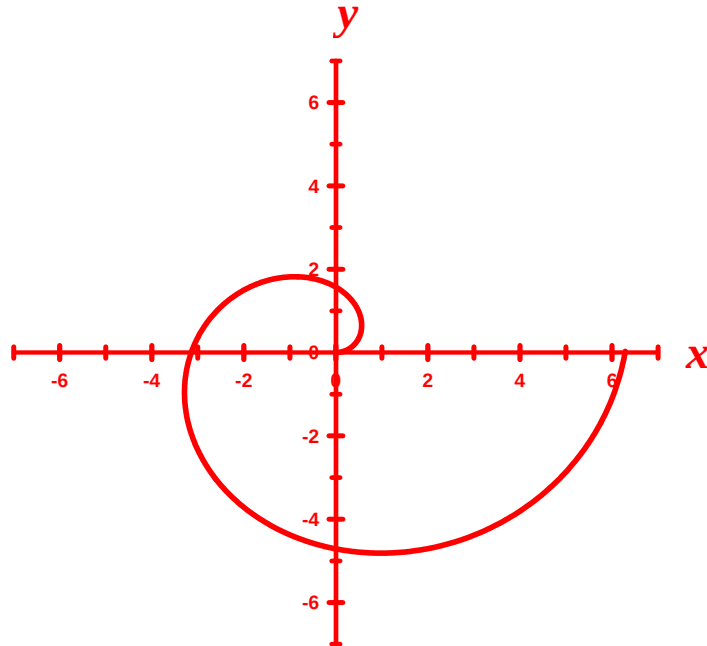
$$L = \int_0^{2\pi} \sqrt{(-\theta \sin \theta + \cos \theta)^2 + (\theta \cos \theta + \sin \theta)^2} d\theta$$

$$L = \int_0^{2\pi} \sqrt{1 + \theta^2} d\theta$$

Lengths & Areas in Polar Coordinates

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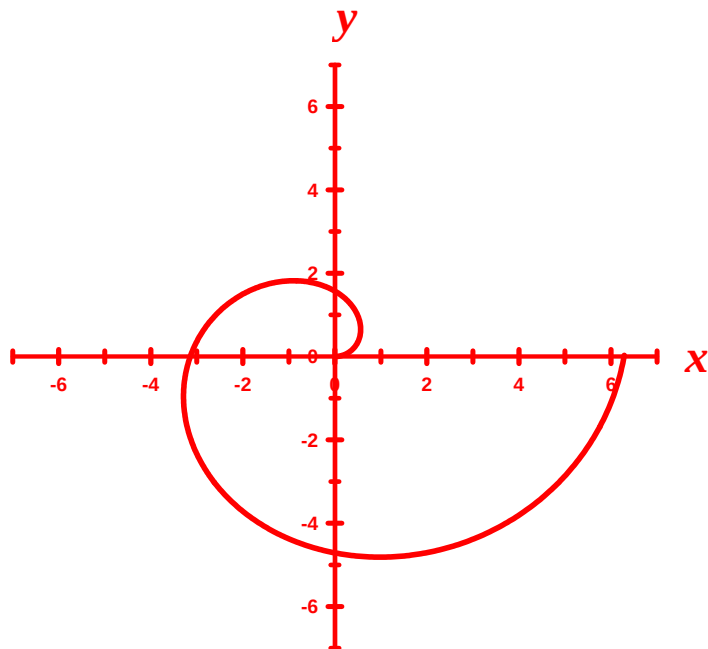
$$L = \int_0^{2\pi} \sqrt{1 + \theta^2} d\theta = ?$$

Fundamental Theorem
or numerical approximation

Lengths & Areas in Polar Coordinates

Problem

What is the length of the section $r = \theta$, $0 \leq \theta \leq 2\pi$ of the spiral ?



Parametric representation:

$$x = \theta \cos \theta, \quad y = \theta \sin \theta$$

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$$L = \int_0^{2\pi} \sqrt{1 + \theta^2} \, d\theta$$

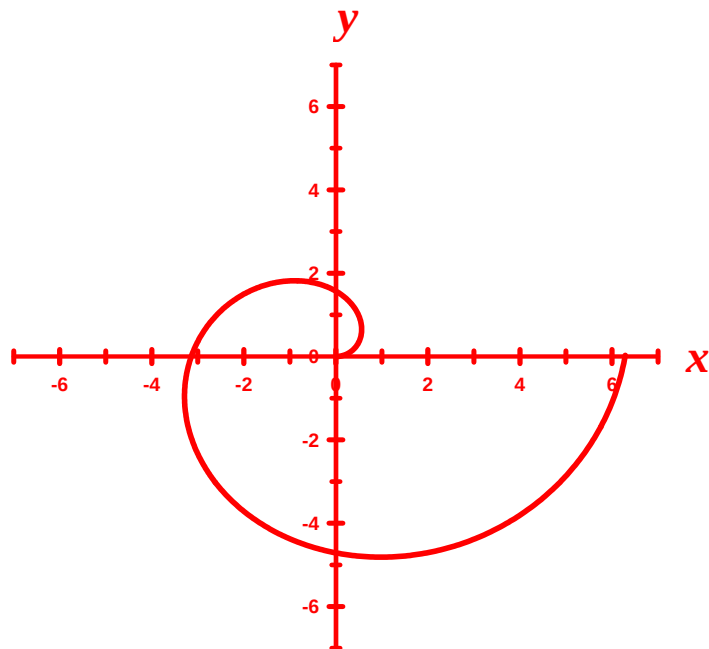
From Table VI, #30 and #28:

$$L = \frac{1}{2} \left(\theta \sqrt{1 + \theta^2} + \ln \left(\theta + \sqrt{1 + \theta^2} \right) \right) \Bigg|_0^{2\pi}$$

Lengths & Areas in Polar Coordinates

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What is the length of the section $r = \theta$, $0 \leq \theta \leq 2\pi$ of the spiral ?



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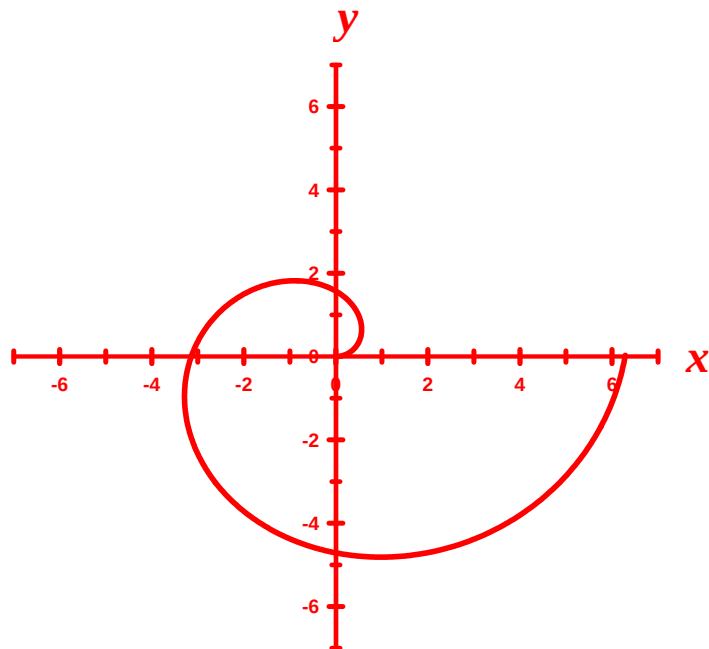
$$L = \frac{1}{2} \left(\theta \sqrt{1 + \theta^2} + \ln \left(\theta + \sqrt{1 + \theta^2} \right) \right) \Bigg|_0^{2\pi}$$

$$L = \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln \left(2\pi + \sqrt{1 + 4\pi^2} \right)$$

Lengths & Areas in Polar Coordinates

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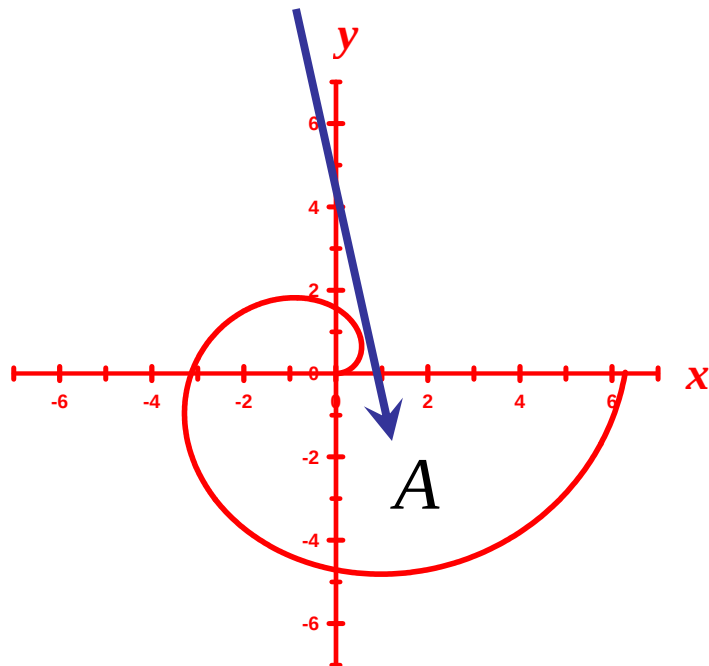
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$$L = \pi \sqrt{1 + 4\pi^2} + \frac{1}{2} \ln \left(2\pi + \sqrt{1 + 4\pi^2} \right) \approx 21.26$$

Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x -axis.

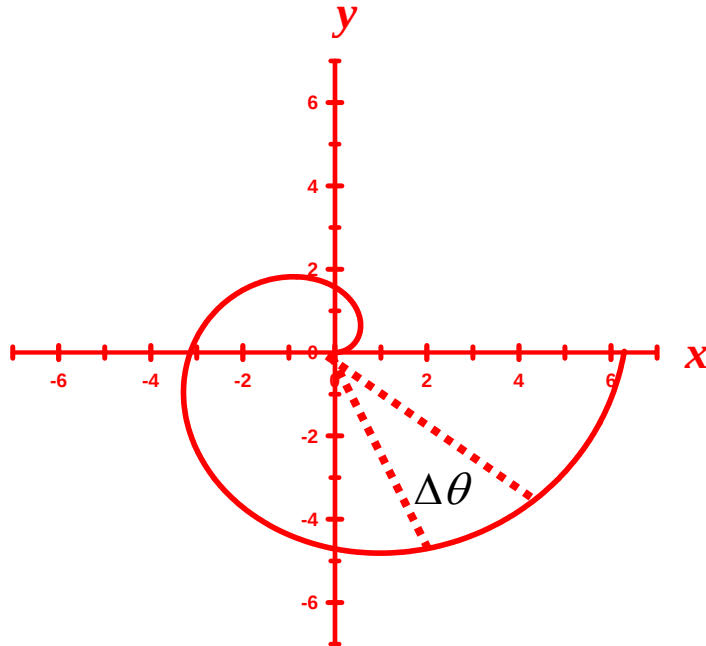


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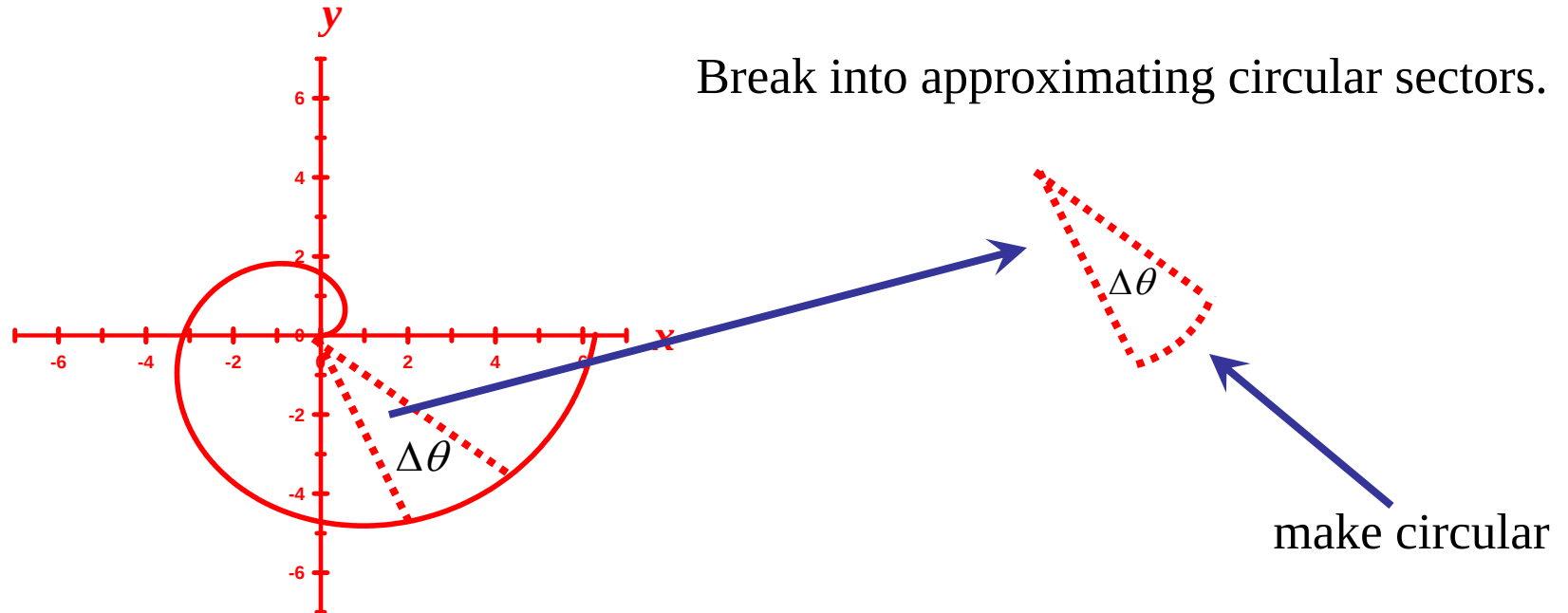
Break into approximating circular sectors.



Lengths & Areas in Polar Coordinates

Problem

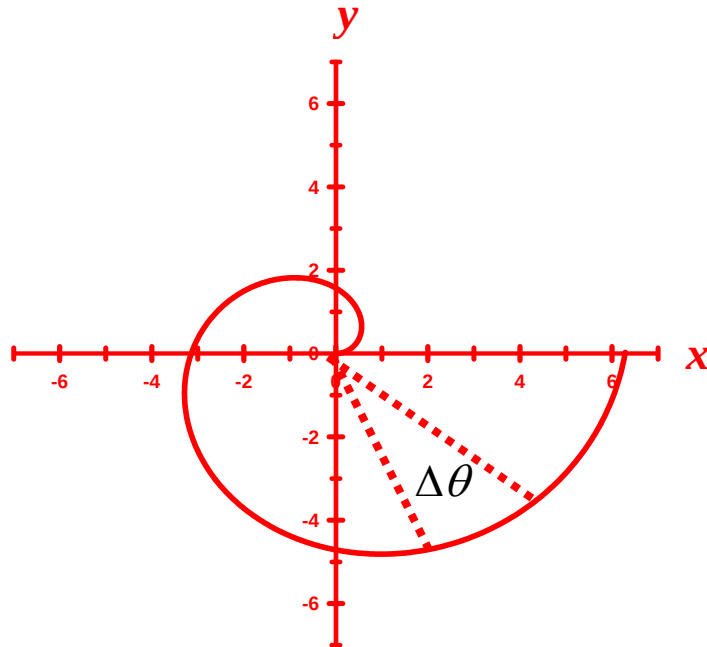
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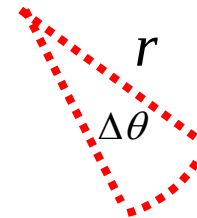
Lengths & Areas in Polar Coordinates

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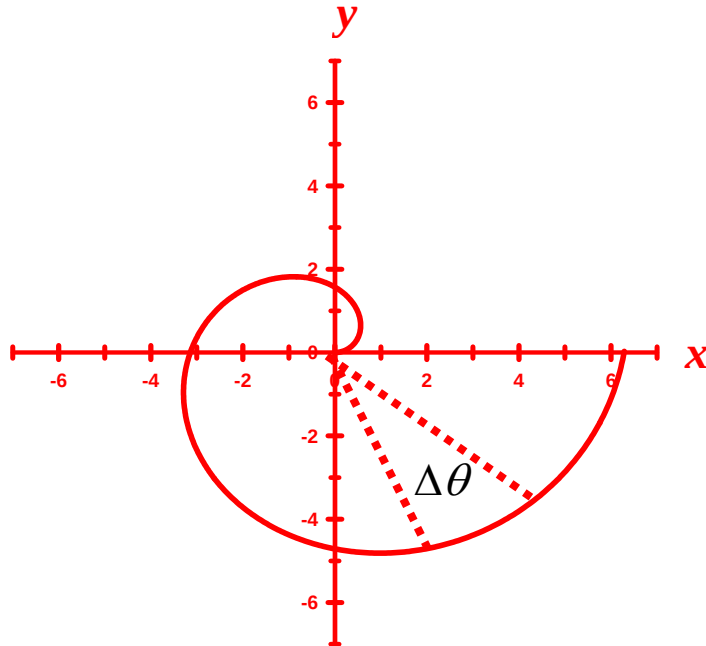


Area of a circular sector = ?

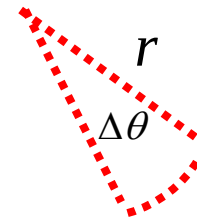
Lengths & Areas in Polar Coordinates

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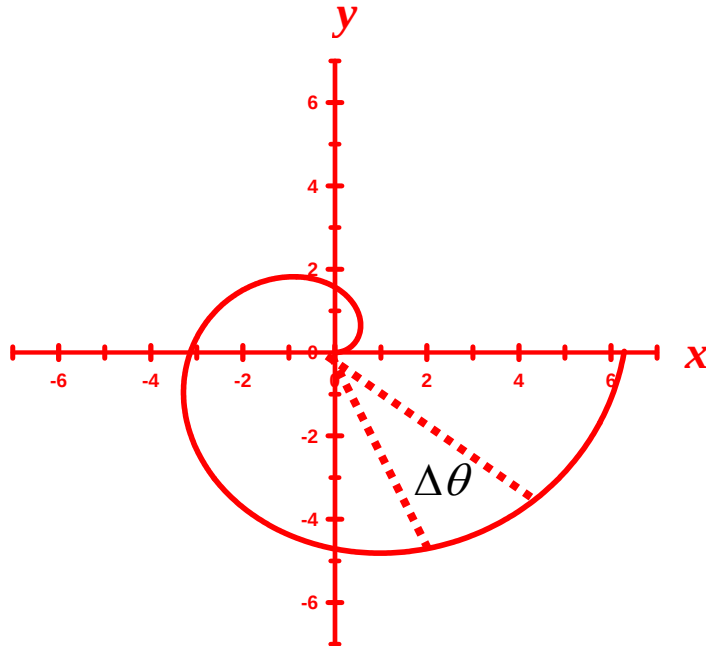
Area of a circular sector = ?

It is a fraction of the area πr^2 of the circle.
Which fraction ?

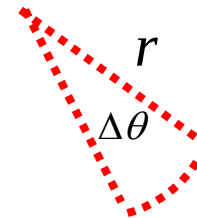
Lengths & Areas in Polar Coordinates

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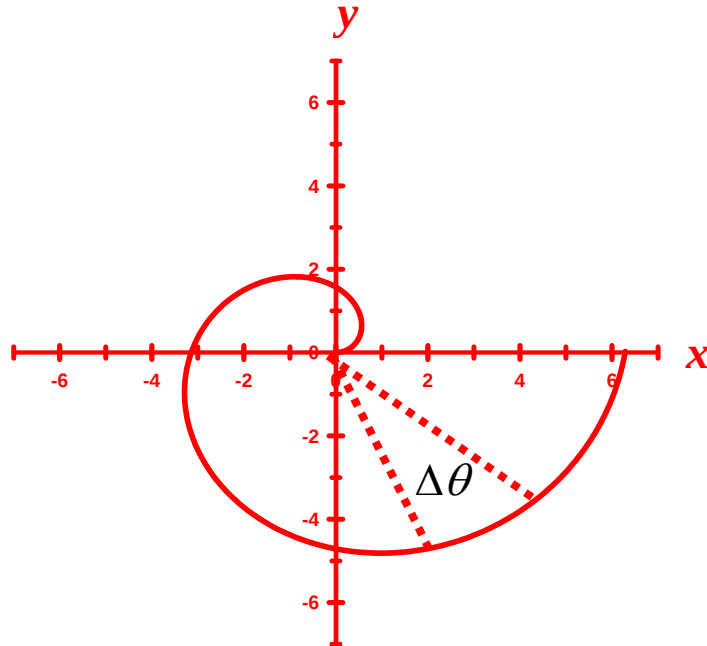
Same fraction that $\Delta\theta$ is of 2π :

$$\frac{\Delta\theta}{2\pi}$$

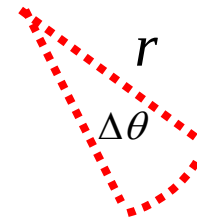
Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x -axis.



Break into approximating circular sectors.



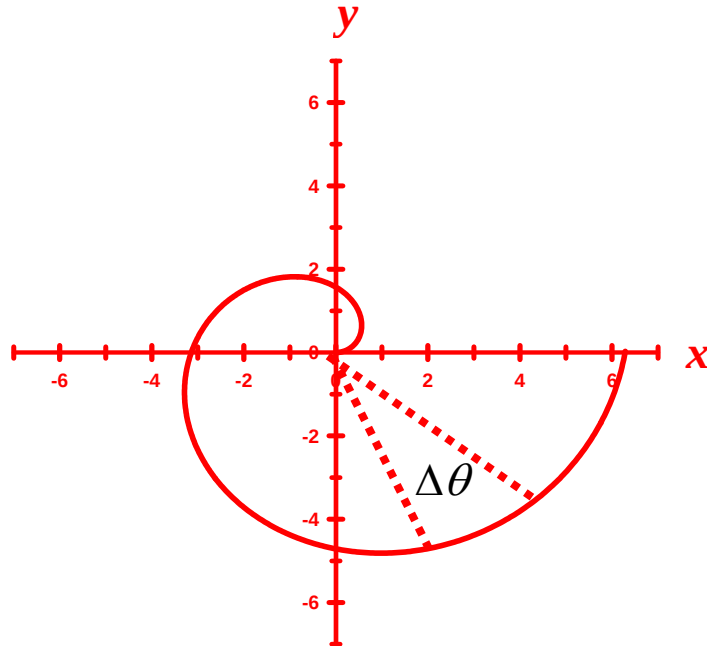
Area of circular sector :

$$\frac{\Delta\theta}{2\pi} \pi r^2$$

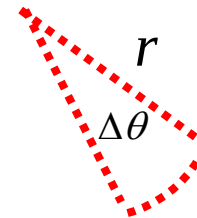
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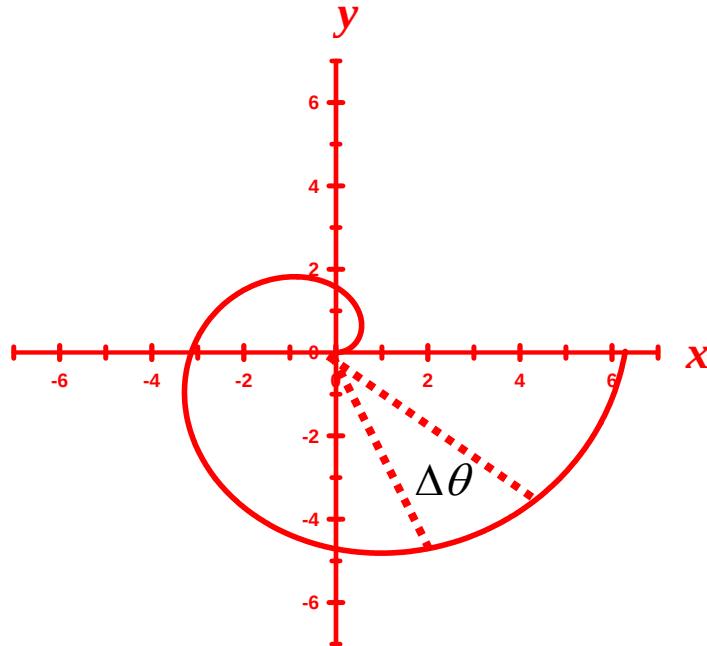
Area of circular sector :

$$\frac{\Delta\theta}{2\pi} \pi r^2 = \frac{1}{2} r^2 \Delta\theta$$

Lengths & Areas in Polar Coordinates

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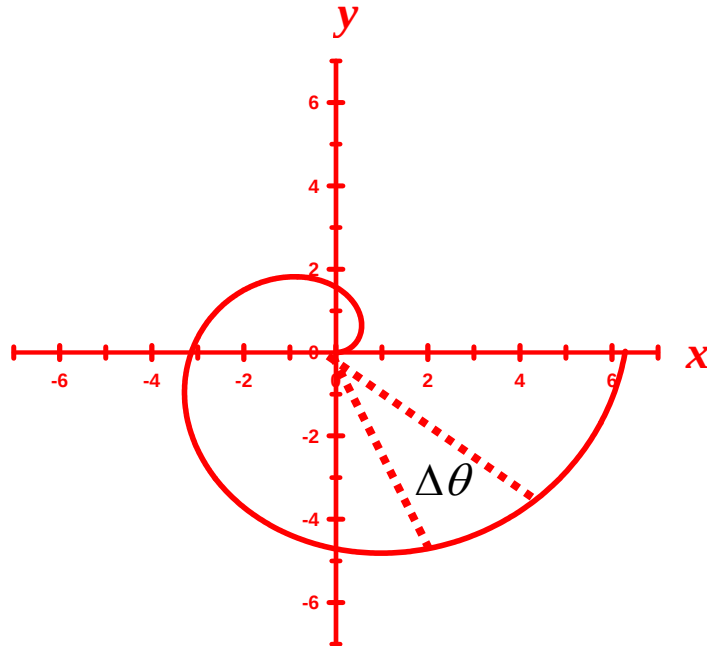
Break into approximating circular sectors.

$$\Delta A \approx \text{Area of circular sector} = \frac{1}{2} r^2 \Delta \theta$$

Lengths & Areas in Polar Coordinates

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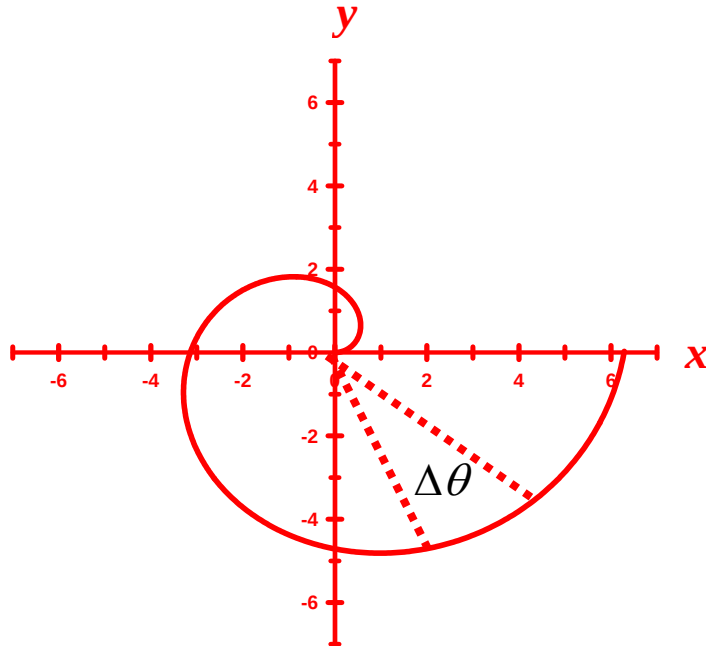
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Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x-axis.



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$$\Delta A \approx \text{Area of circular sector} = \frac{1}{2} r^2 \Delta \theta$$

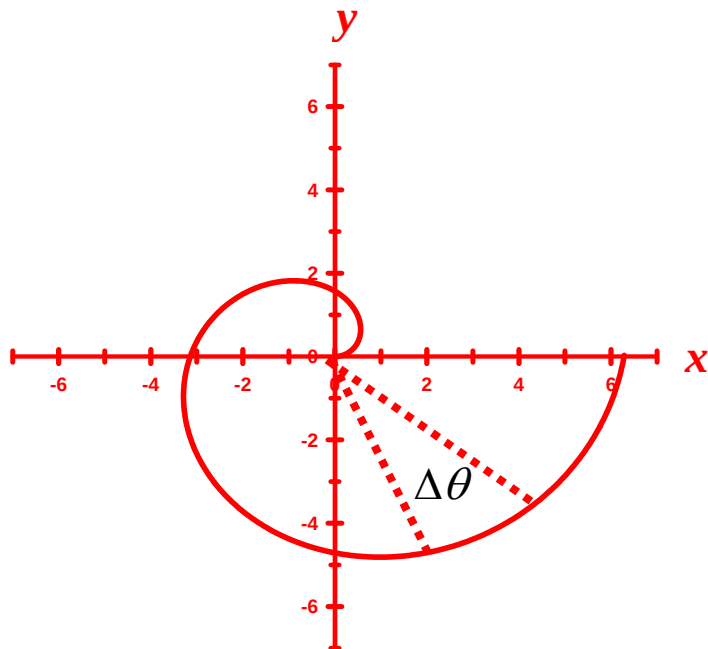
$$\Delta A \approx \frac{1}{2} \theta^2 \Delta \theta$$

$$\text{Total area} = A \approx \sum_{\theta=0}^{2\pi} \frac{1}{2} \theta^2 \Delta \theta$$

Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x-axis.



Break into approximating circular sectors.

$$\Delta A \approx \text{Area of circular sector} = \frac{1}{2} r^2 \Delta \theta$$

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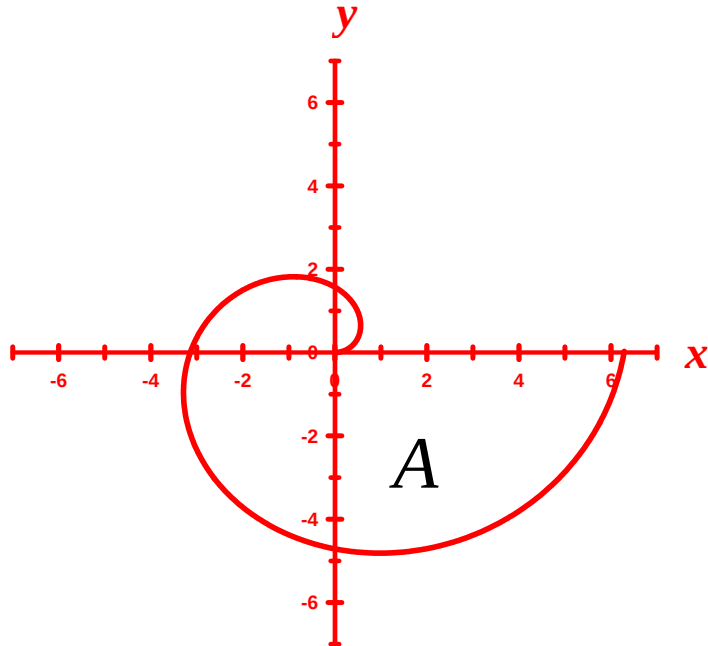
$$\text{Total area} = A \approx \sum_{\theta=0}^{2\pi} \frac{1}{2} \theta^2 \Delta \theta$$

$$\text{Pass to the limit: } A = \int_0^{2\pi} \frac{1}{2} \theta^2 d\theta$$

Lengths & Areas in Polar Coordinates

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Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x-axis.

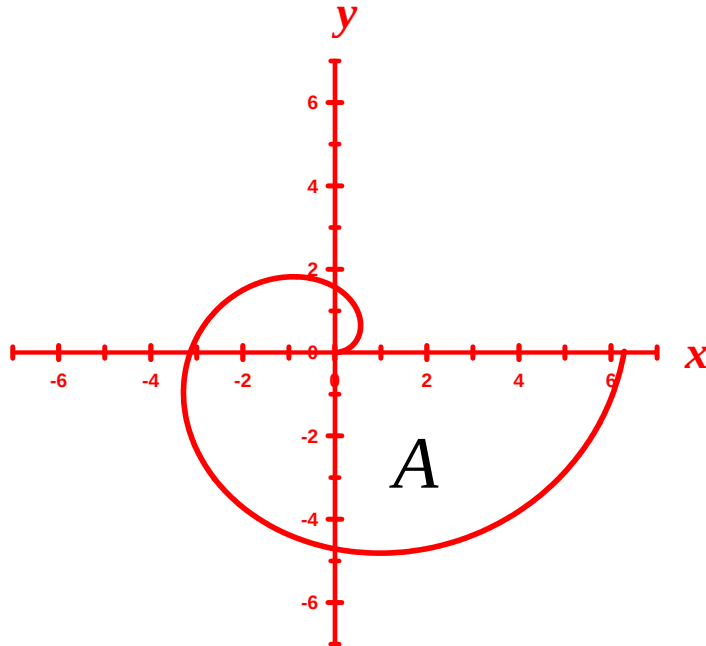


$$A = \int_0^{2\pi} \frac{1}{2} \theta^2 d\theta$$

Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x -axis.

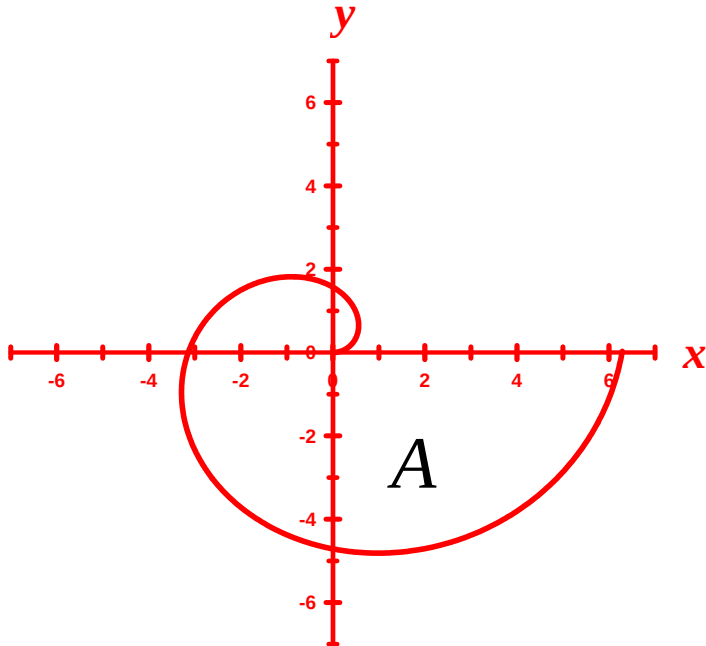


$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} \theta^2 d\theta \\ &= \frac{1}{6} \theta^3 \Big|_0^{2\pi} \end{aligned}$$

Lengths & Areas in Polar Coordinates

Problem

Find the area enclosed by the spiral $r = \theta$, $0 \leq \theta \leq 2\pi$ and the x-axis.



$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} \theta^2 d\theta \\ &= \frac{1}{6} \theta^3 \Big|_0^{2\pi} = \frac{4}{3} \pi^3 \end{aligned}$$

Lengths & Areas in Polar Coordinates

In general, the area enclosed by curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is

$$A = \lim \sum_{\alpha}^{\beta} \left[\frac{1}{2} r^2 \Delta \theta \right]$$

Areas of approximating circular sectors.

Lengths & Areas in Polar Coordinates

In general, the area enclosed by curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is

$$A = \lim \sum_{\alpha}^{\beta} \frac{1}{2} r^2 \Delta \theta$$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta$$