

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, October 16

<http://math.arizona.edu/~cushing/250a.html>

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

For example,

approximating rectangles or trapezoid in the case of areas.

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

Sum up approximating pieces.

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

Sum up approximating pieces.

Pass to a limit (number of pieces goes to infinity).

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

Sum up approximating pieces.

Pass to a limit (number of pieces goes to infinity).

Approximating sum will approach a definite integral.

Chapter 8: Using the Definite Integral

We'll look at selected applications from this chapter.

General theme:

Break problem into approximating pieces of simpler, tractable form.

Sum up approximating pieces.

Pass to a limit (number of pieces goes to infinity).

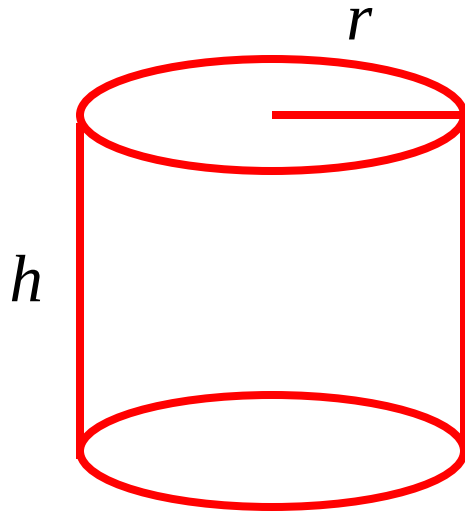
Approximating sum will approach a definite integral.

Calculate the definite integral to get exact answer.

Volumes of Solids

Volumes of Solids

What's the volume of a cylinder ?

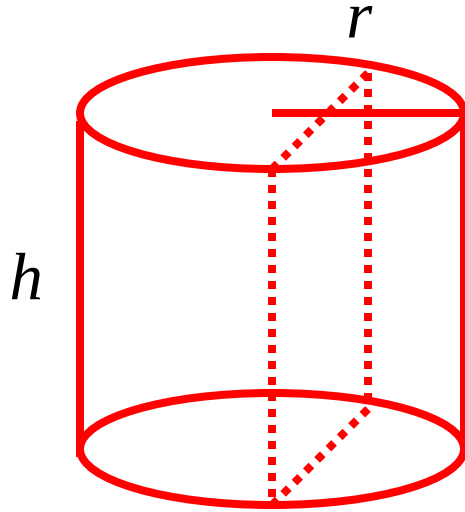


r = radius

h = height

Volumes of Solids

What's the volume of a cylinder ?



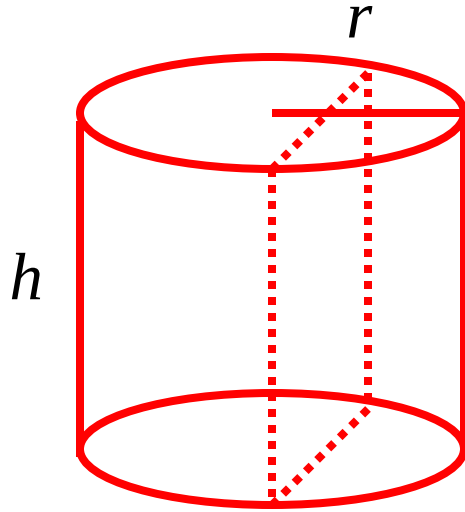
r = radius

h = height

Slice into thin, vertical slabs
each of which is (approximately)
a flat box.

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

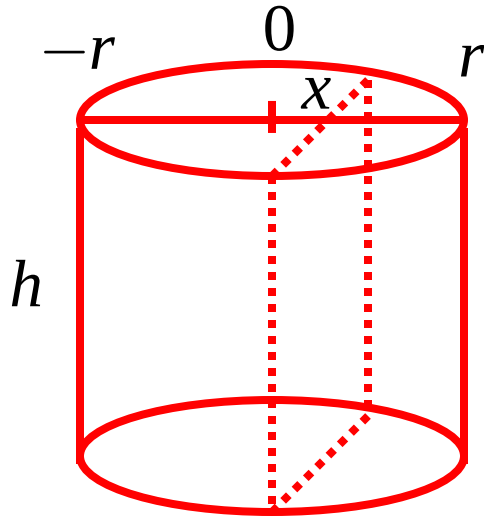
Slice into thin, vertical slabs
each of which is (approximately)
a flat box.

volume is

base $\times$ height $\times$ thickness

Volumes of Solids

What's the volume of a cylinder ?



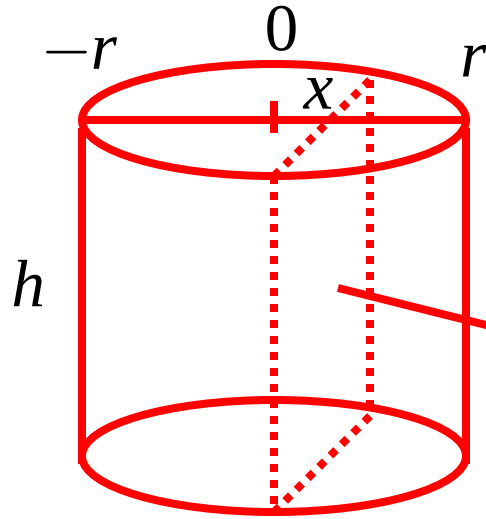
a typical slice is at position
 x between $-r$ and r

$r = \text{radius}$

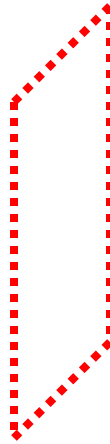
$h = \text{height}$

Volumes of Solids

What's the volume of a cylinder ?



a typical slice is at position
 x between $-r$ and r

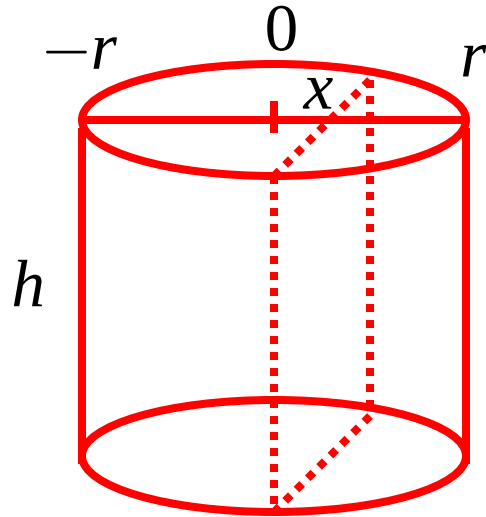


$r = \text{radius}$

$h = \text{height}$

Volumes of Solids

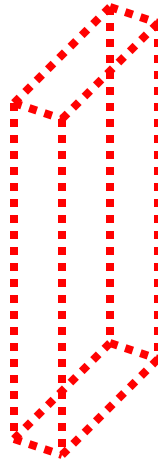
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position
 x between $-r$ and r

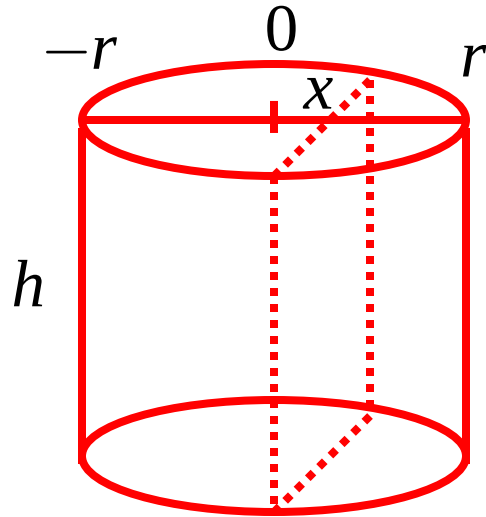


Magnify
& square off edges
to get an approximating box

What is it's volume ?

Volumes of Solids

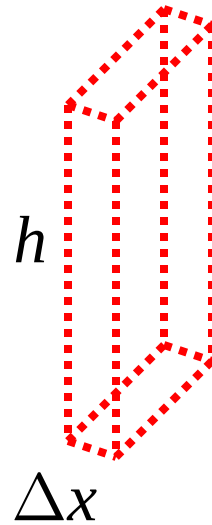
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position
 x between $-r$ and r

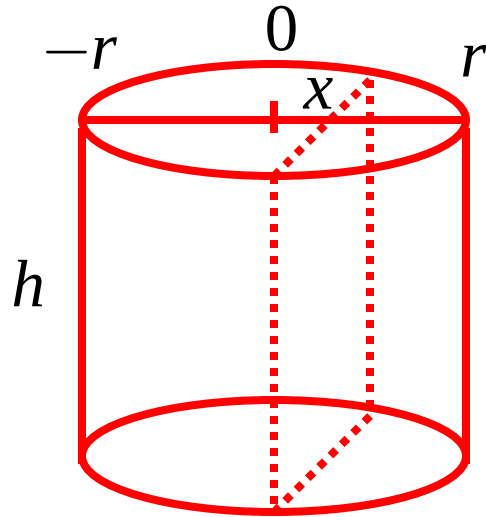


Magnify
& square off edges
to get an approximating box

What is it's volume ?

Volumes of Solids

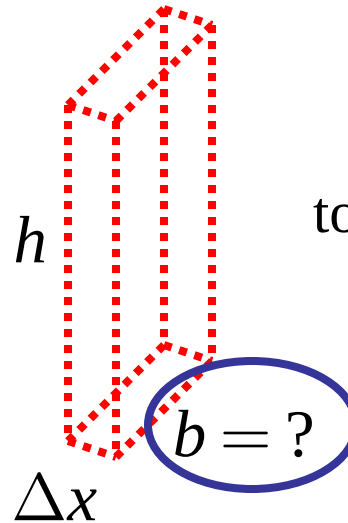
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position
 x between $-r$ and r

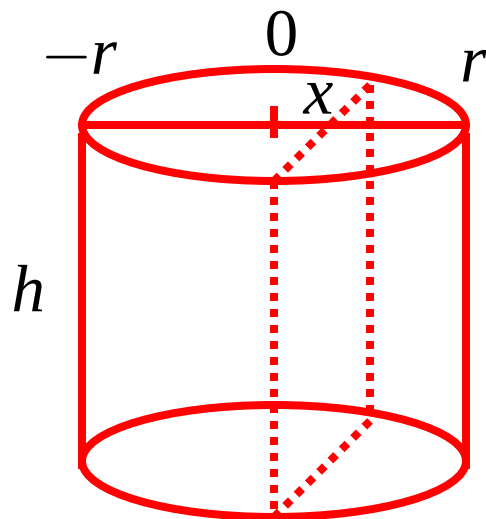


Magnify
& square off edges
to get an approximating box

What is it's volume ?

Volumes of Solids

What's the volume of a cylinder ?

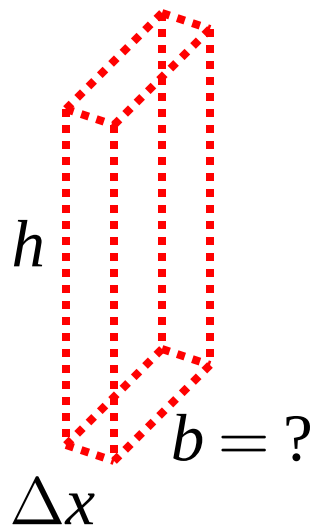


$r = \text{radius}$

$h = \text{height}$

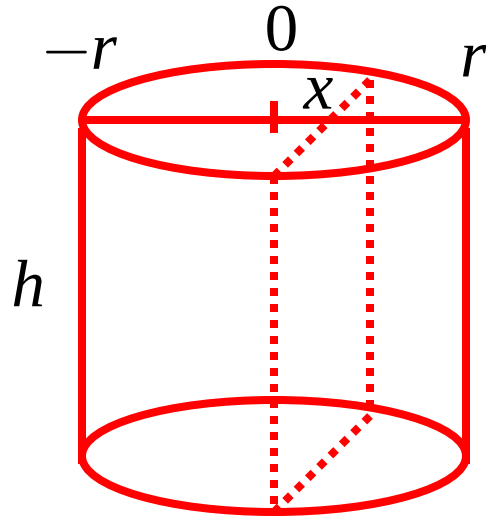
a typical slice is at position
 x between $-r$ and r

base b depends
on slicing position x



Volumes of Solids

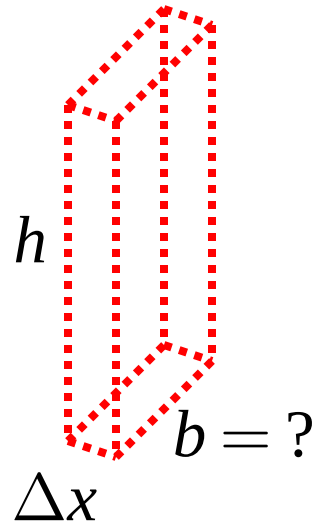
What's the volume of a cylinder ?



$r = \text{radius}$

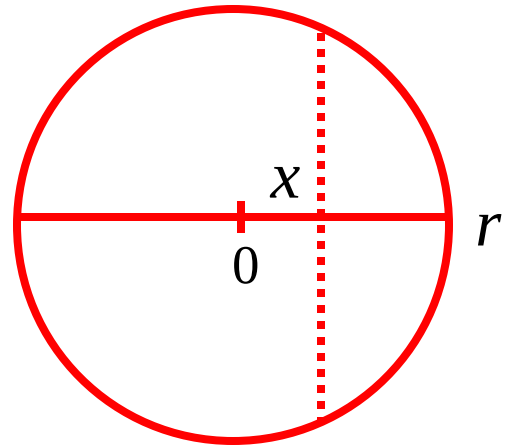
$h = \text{height}$

a typical slice is at position
 x between $-r$ and r



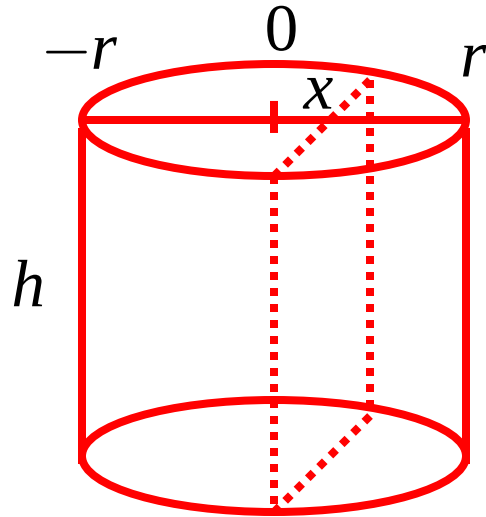
base b depends
on slicing position x

Look down from top:



Volumes of Solids

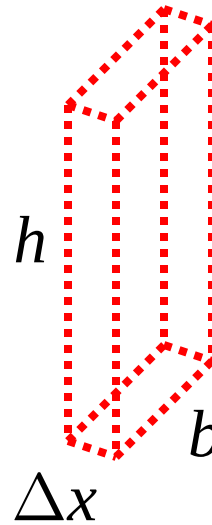
What's the volume of a cylinder ?



$r = \text{radius}$

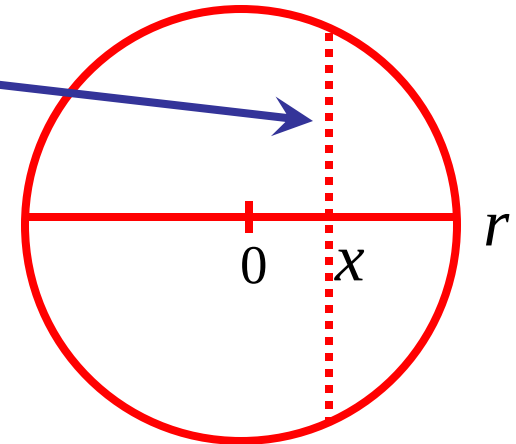
$h = \text{height}$

a typical slice is at position x between $-r$ and r



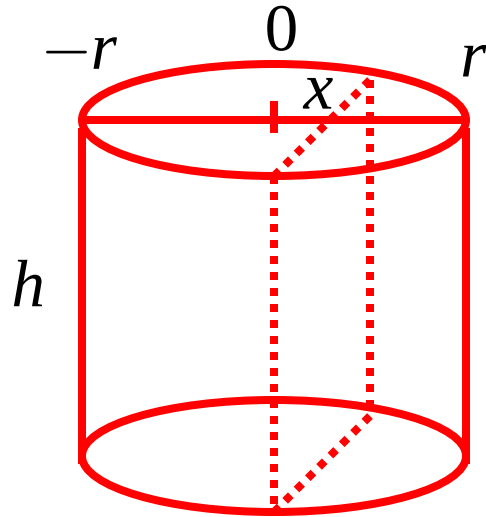
base b depends
on slicing position x

Look down from top:



Volumes of Solids

What's the volume of a cylinder ?

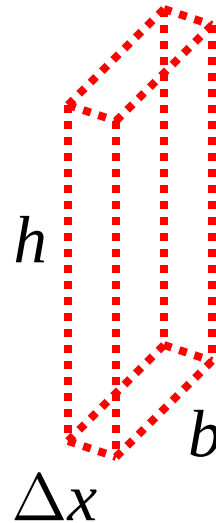


$r = \text{radius}$

$h = \text{height}$

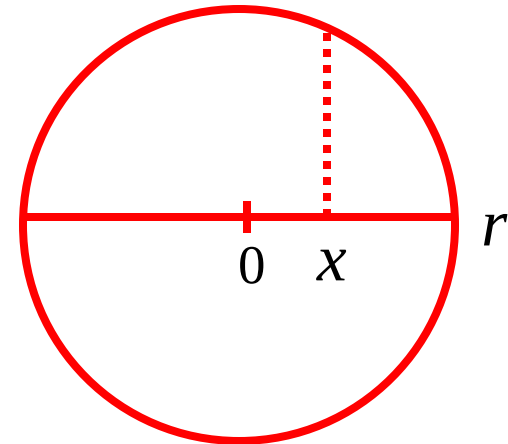
Equation for circle: $x^2 + y^2 = r^2$

a typical slice is at position x between $-r$ and r



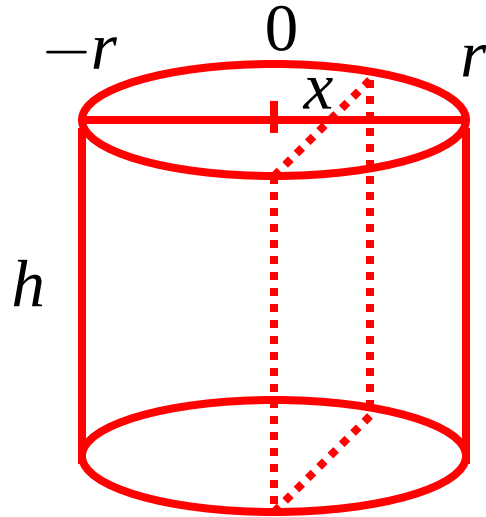
base b depends
on slicing position x

Look down from top:



Volumes of Solids

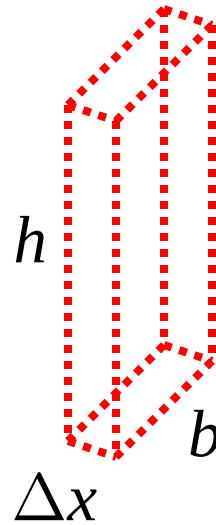
What's the volume of a cylinder ?



$r = \text{radius}$

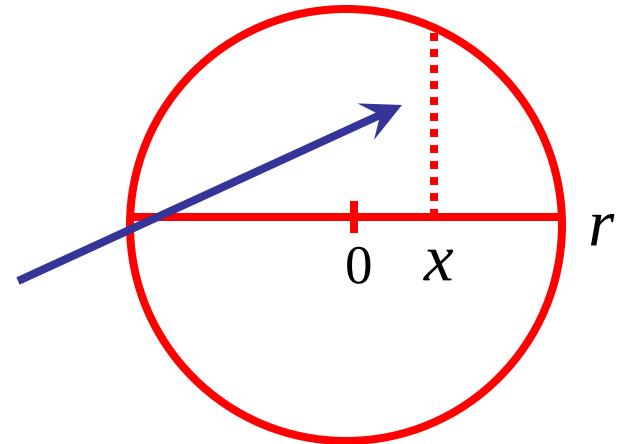
$h = \text{height}$

a typical slice is at position x between $-r$ and r



base b depends
on slicing position x

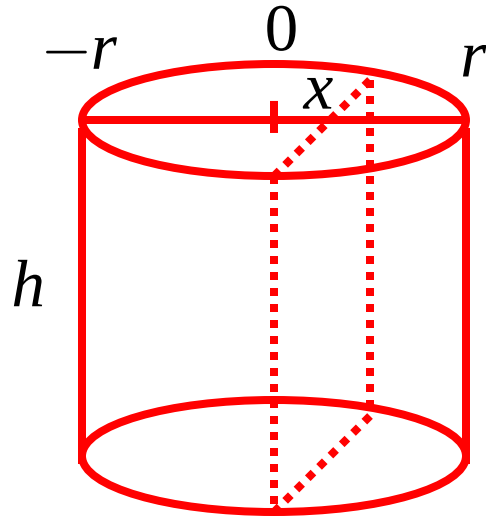
Look down from top:



Equation for circle: $y = \sqrt{r^2 - x^2}$

Volumes of Solids

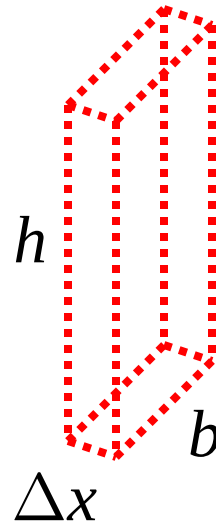
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

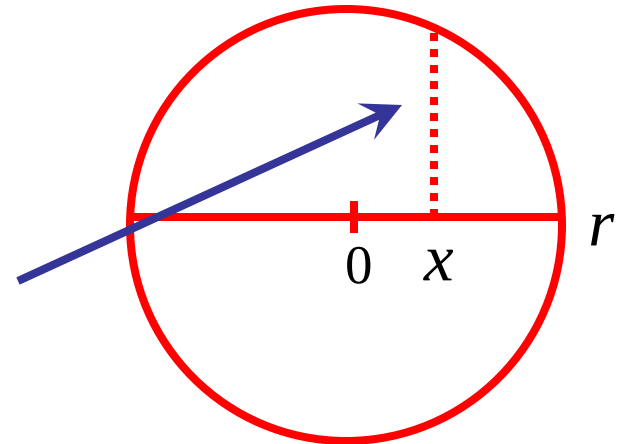
a typical slice is at position x between $-r$ and r



Base is

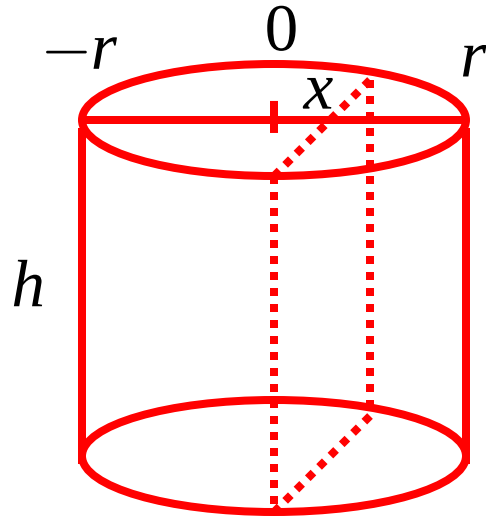
$$b = 2\sqrt{r^2 - x^2}$$

Equation for circle: $y = \sqrt{r^2 - x^2}$



Volumes of Solids

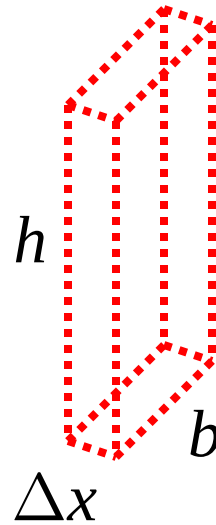
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position
 x between $-r$ and r

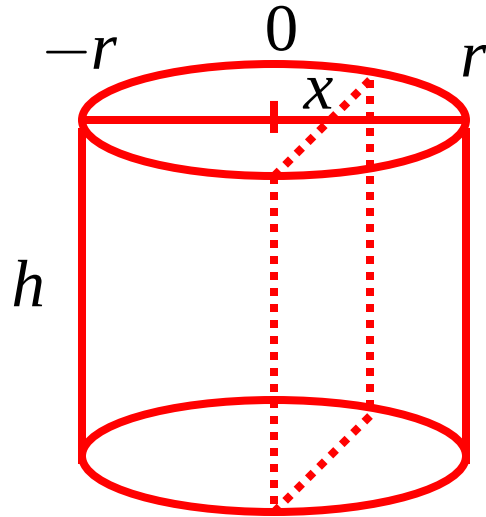


Volume of approximating box:

$$\Delta V = 2\sqrt{r^2 - x^2} \times h \times \Delta x$$

Volumes of Solids

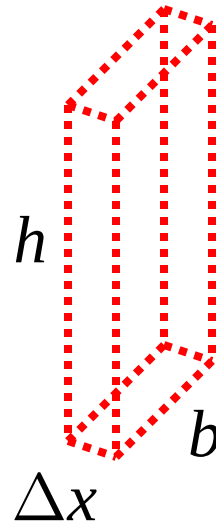
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position
 x between $-r$ and r



Volume of approximating box:

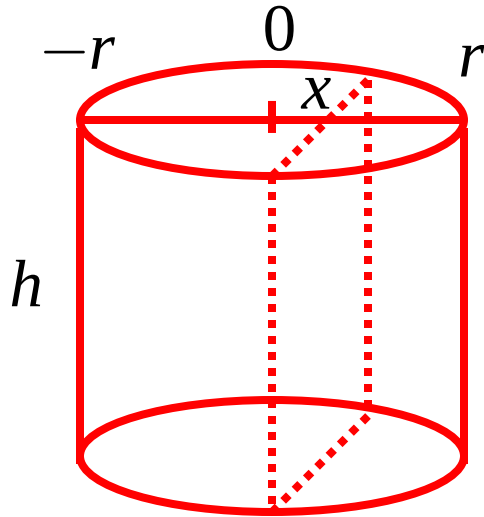
$$\Delta V = 2\sqrt{r^2 - x^2} \times h \times \Delta x$$

Sum of approximating boxes:

$$V \approx \sum 2\sqrt{r^2 - x^2} h \Delta x$$

Volumes of Solids

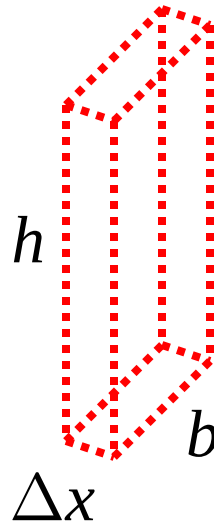
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position x between $-r$ and r



Volume of approximating box:

$$\Delta V = 2\sqrt{r^2 - x^2} \times h \times \Delta x$$

Sum of approximating boxes:

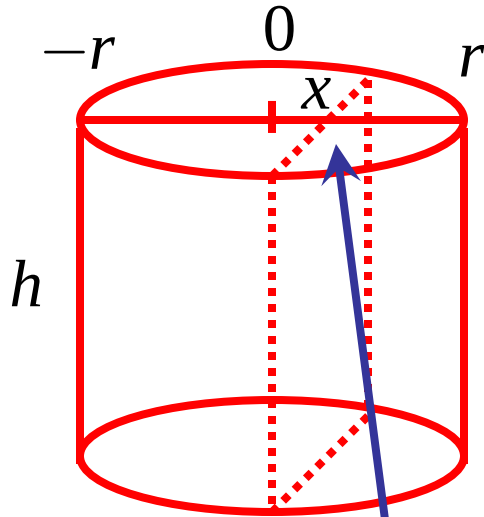
$$V \approx \sum 2\sqrt{r^2 - x^2} h \Delta x$$

Pass number of slices to ∞ :

$$V = \int_{-r}^r 2\sqrt{r^2 - x^2} h dx$$

Volumes of Solids

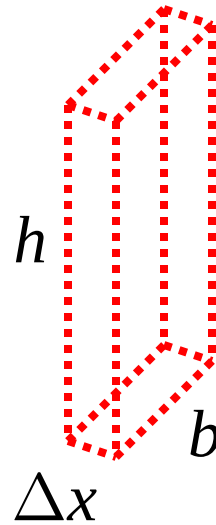
What's the volume of a cylinder ?



$r = \text{radius}$

$h = \text{height}$

a typical slice is at position x between $-r$ and r



Volume of approximating box:

$$\Delta V = 2\sqrt{r^2 - x^2} \times h \times \Delta x$$

Sum of approximating boxes:

$$V \approx \sum 2\sqrt{r^2 - x^2} h \Delta x$$

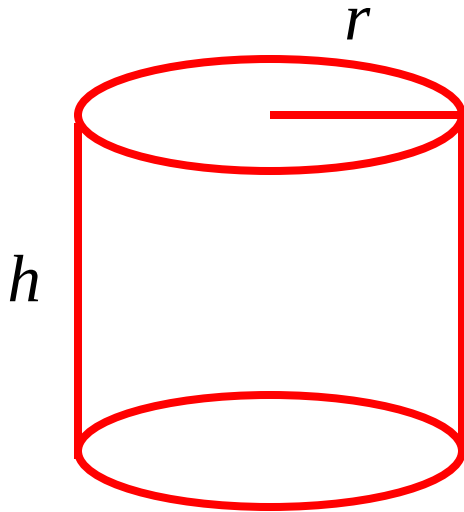
Pass number of slices to ∞ :

$$V = \int_{-r}^r 2\sqrt{r^2 - x^2} h dx$$

because slices are made
between $-r$ and r

Volumes of Solids

What's the volume of a cylinder ?



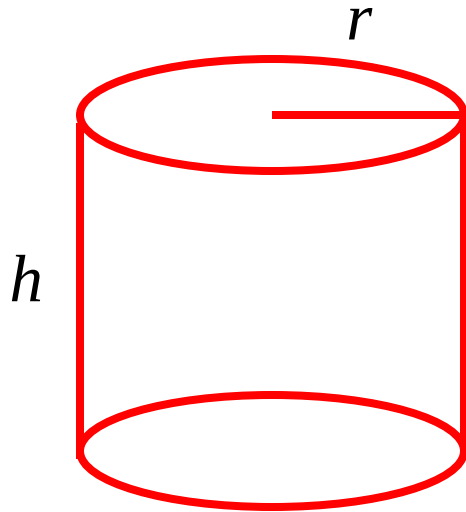
r = radius

h = height

$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

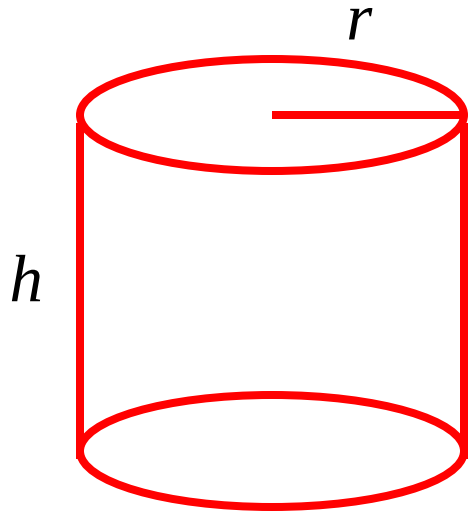
$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$\int \sqrt{r^2 - x^2} dx = \frac{1}{2} \left(x\sqrt{r^2 - x^2} + r^2 \int \frac{1}{\sqrt{r^2 - x^2}} dx \right)$$

Table VI, #30

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

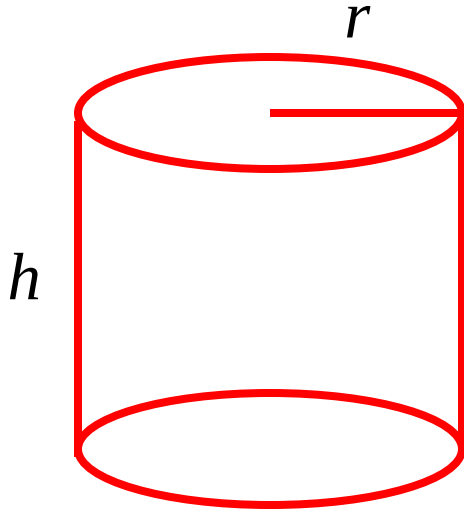
$$\int \sqrt{r^2 - x^2} dx = \frac{1}{2} \left(x\sqrt{r^2 - x^2} + r^2 \int \frac{1}{\sqrt{r^2 - x^2}} dx \right)$$

$$= \frac{1}{2} \left(x\sqrt{r^2 - x^2} + r^2 \arcsin \left(\frac{x}{r} \right) \right)$$

Table VI, #28

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

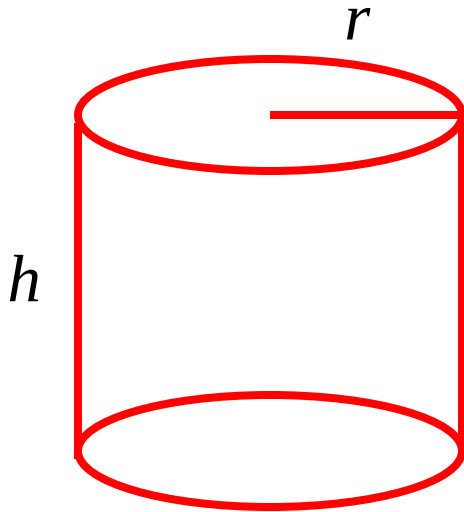
$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$V = 2h \frac{1}{2} \left(x\sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \bigg|_{x=-r}^{x=r}$$

Fundamental Theorem

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

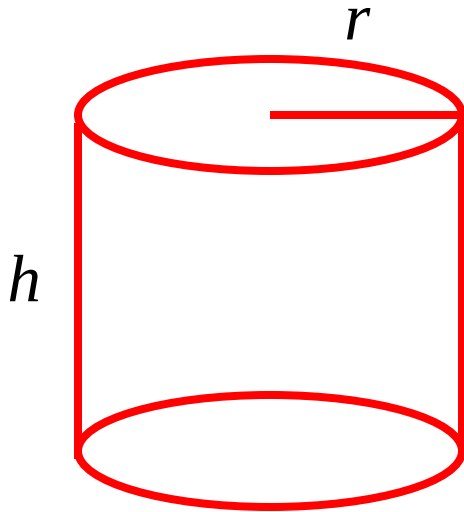
$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$V = 2h \frac{1}{2} \left(x\sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \Bigg|_{x=-r}^{x=r}$$

$$V = 2h \left(\frac{1}{2} r^2 \arcsin 1 - \frac{1}{2} r^2 \arcsin(-1) \right)$$

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

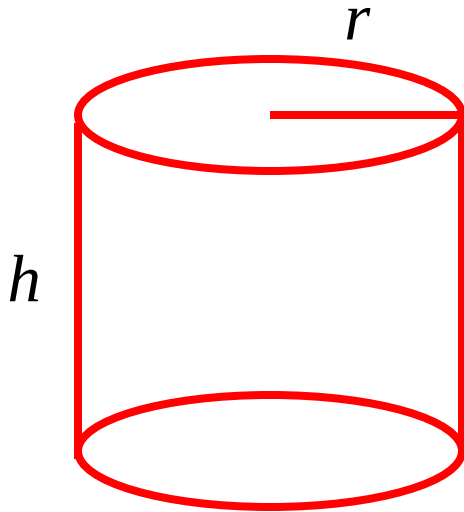
$$V = 2h \frac{1}{2} \left(x \sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \Bigg|_{x=-r}^{x=r}$$

$$V = 2h \left(\frac{1}{2} r^2 \arcsin 1 - \frac{1}{2} r^2 \arcsin(-1) \right)$$

$$V = h \left(r^2 \frac{\pi}{2} - r^2 \left(-\frac{\pi}{2} \right) \right)$$

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

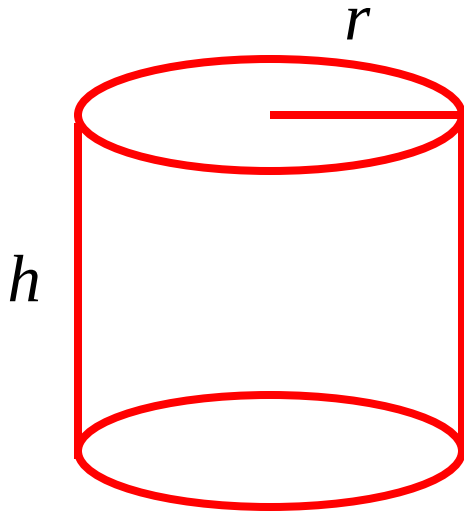
$$V = 2h \frac{1}{2} \left(x \sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \Bigg|_{x=-r}^{x=r}$$

$$V = 2h \left(\frac{1}{2} r^2 \arcsin 1 - \frac{1}{2} r^2 \arcsin(-1) \right)$$

$$V = h \left(r^2 \frac{\pi}{2} - r^2 \left(-\frac{\pi}{2} \right) \right) = \pi r^2 h$$

Volumes of Solids

What's the volume of a cylinder ?



r = radius

h = height

$$V = 2h \int_{-r}^r \sqrt{r^2 - x^2} dx$$

$$V = 2h \frac{1}{2} \left(x \sqrt{r^2 - x^2} + r^2 \arcsin \frac{x}{r} \right) \Bigg|_{x=-r}^{x=r}$$

$$V = 2h \left(\frac{1}{2} r^2 \arcsin 1 - \frac{1}{2} r^2 \arcsin(-1) \right)$$

$$V = h \left(r^2 \frac{\pi}{2} - r^2 \left(-\frac{\pi}{2} \right) \right) = \pi r^2 h$$

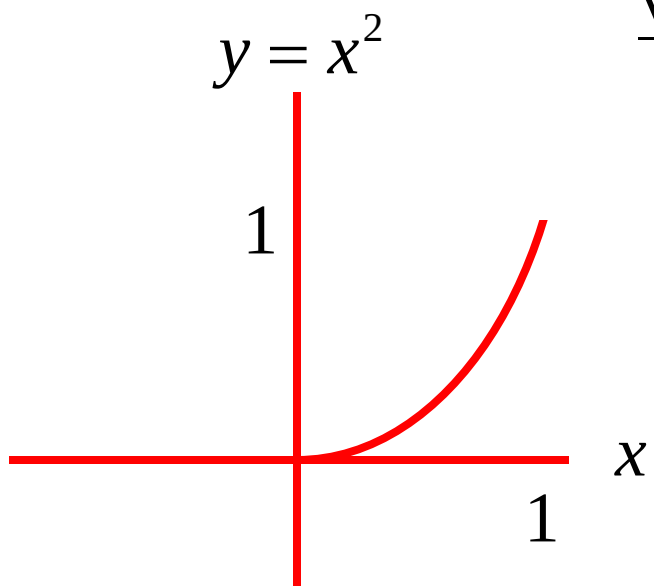
area of base x height

Volumes of Solids

We can now use cylinders as approximating solids.

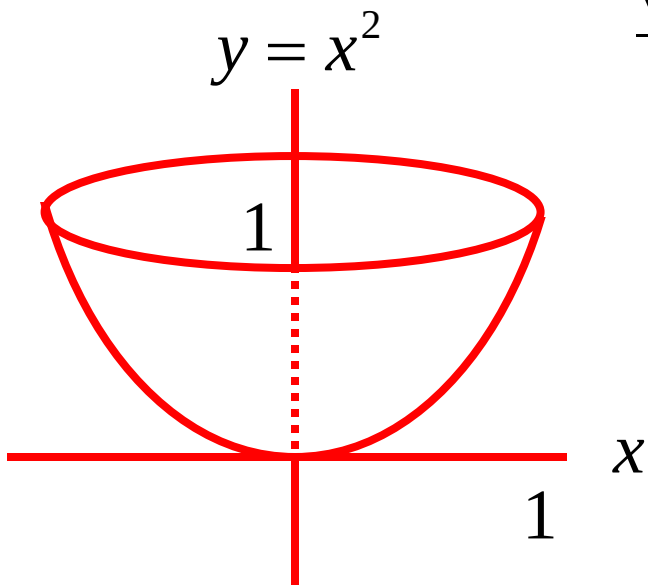
This is done, for example, with *solids of revolution*.

Volumes of Solids



Volumes of Solids

What's the volume V ?

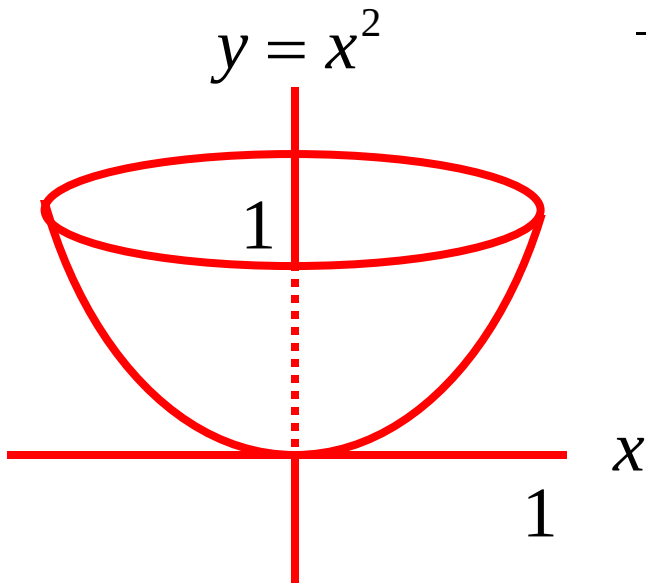


Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation

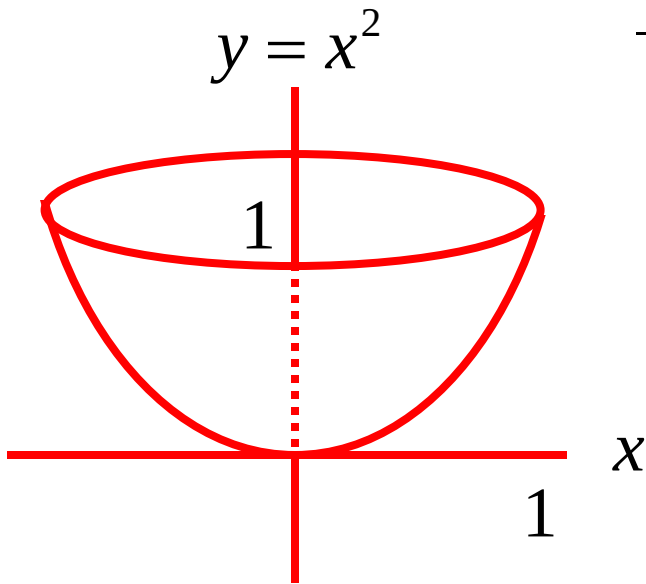


Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
*slicing perpendicular to the axis of
rotation*



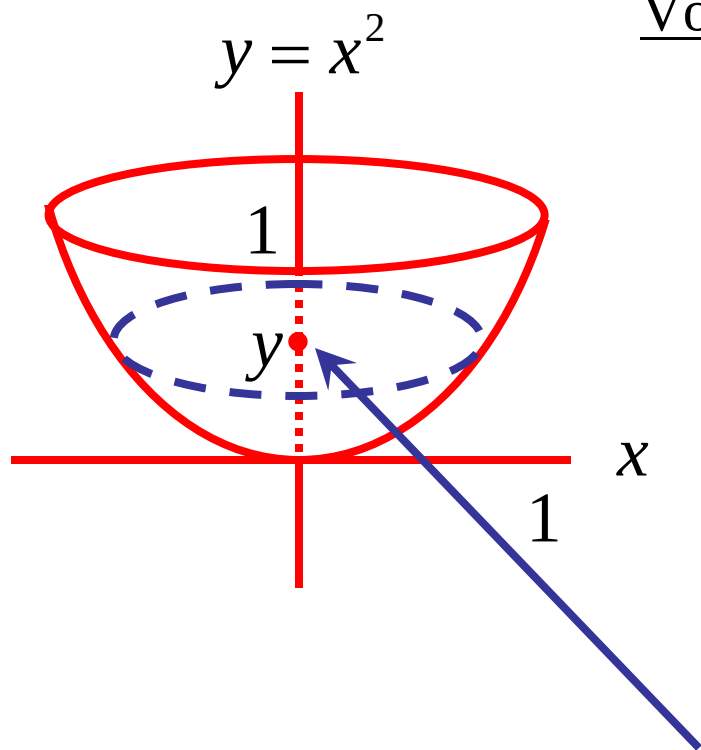
A general rule of thumb
for solids of revolution

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation



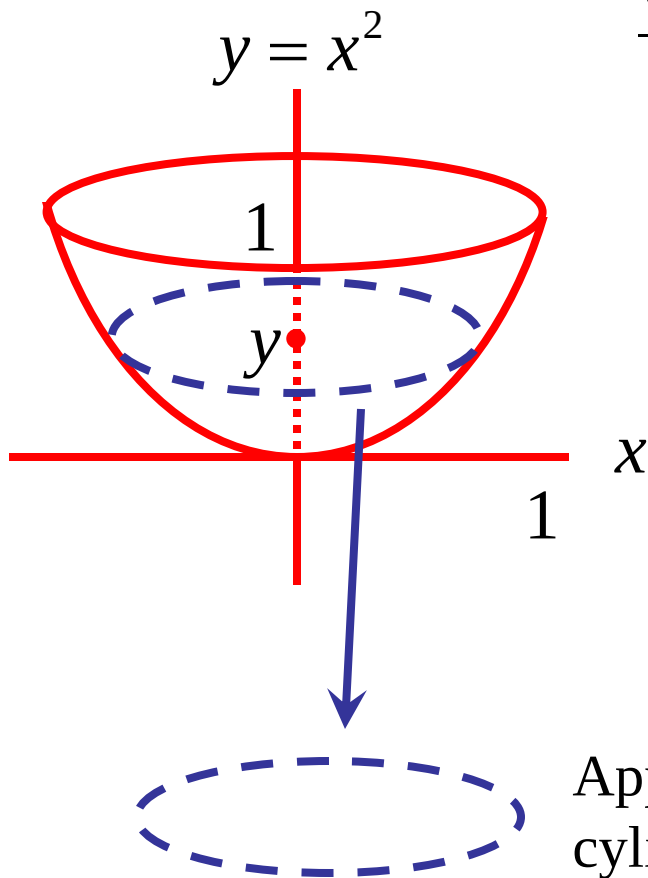
slice at a level y between 0 and 1

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by slicing perpendicular to the axis of rotation



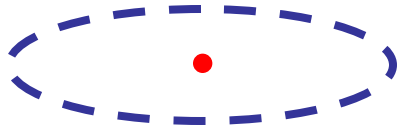
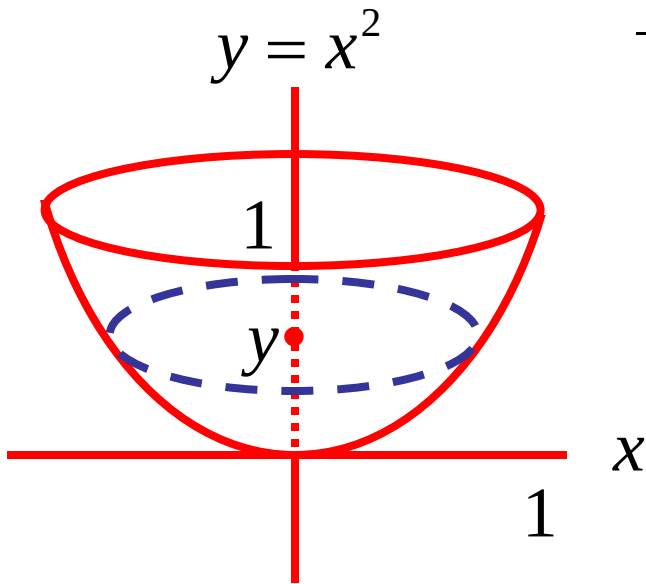
Approximately a flat cylinder of height Δy

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation



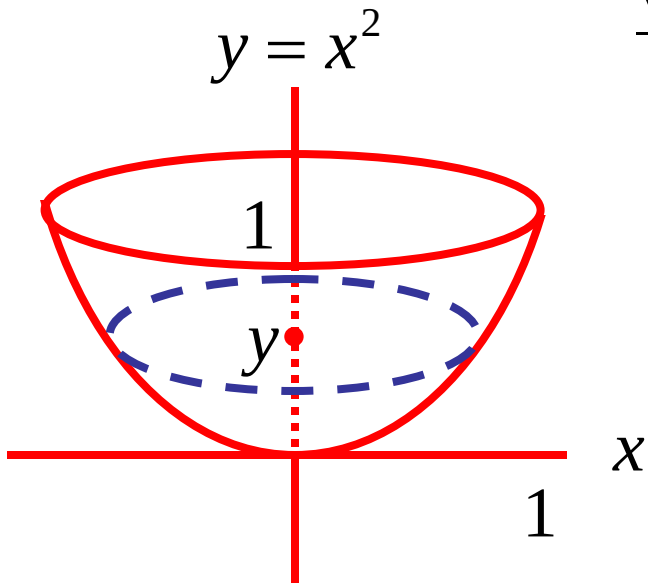
Or approximately a
disk of thickness Δy

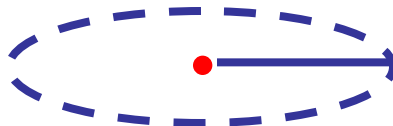
Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation



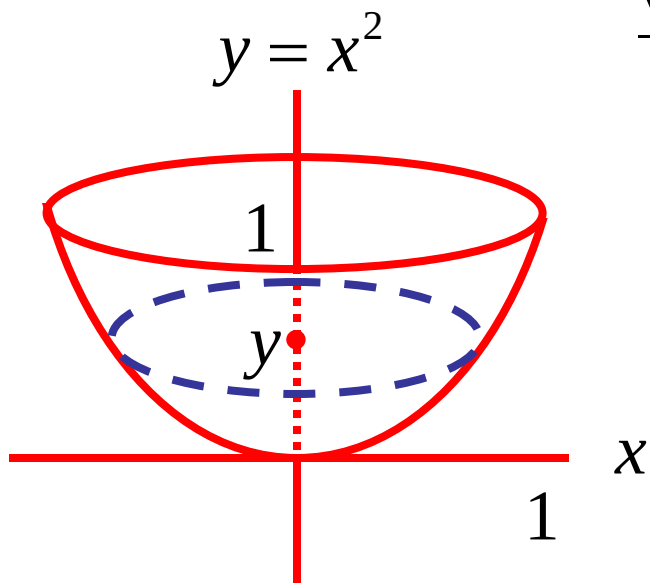

$$\Delta V \approx \pi r^2 \Delta y$$

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation



$$\Delta V \approx \pi r^2 \Delta y$$

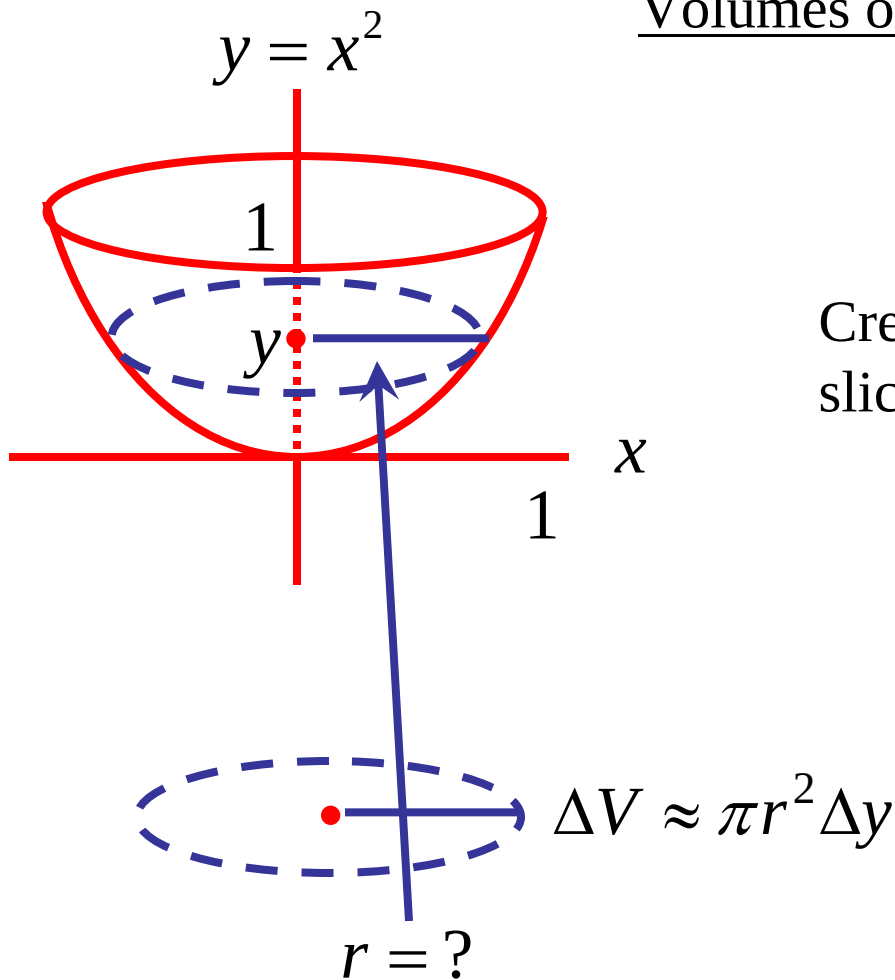
$r = ?$

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by slicing perpendicular to the axis of rotation

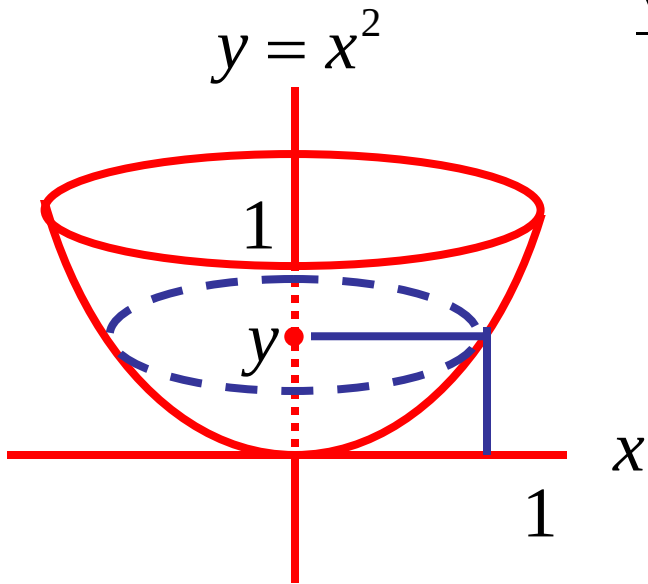


Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by
slicing perpendicular to the axis of
rotation



$\Delta V \approx \pi r^2 \Delta y$

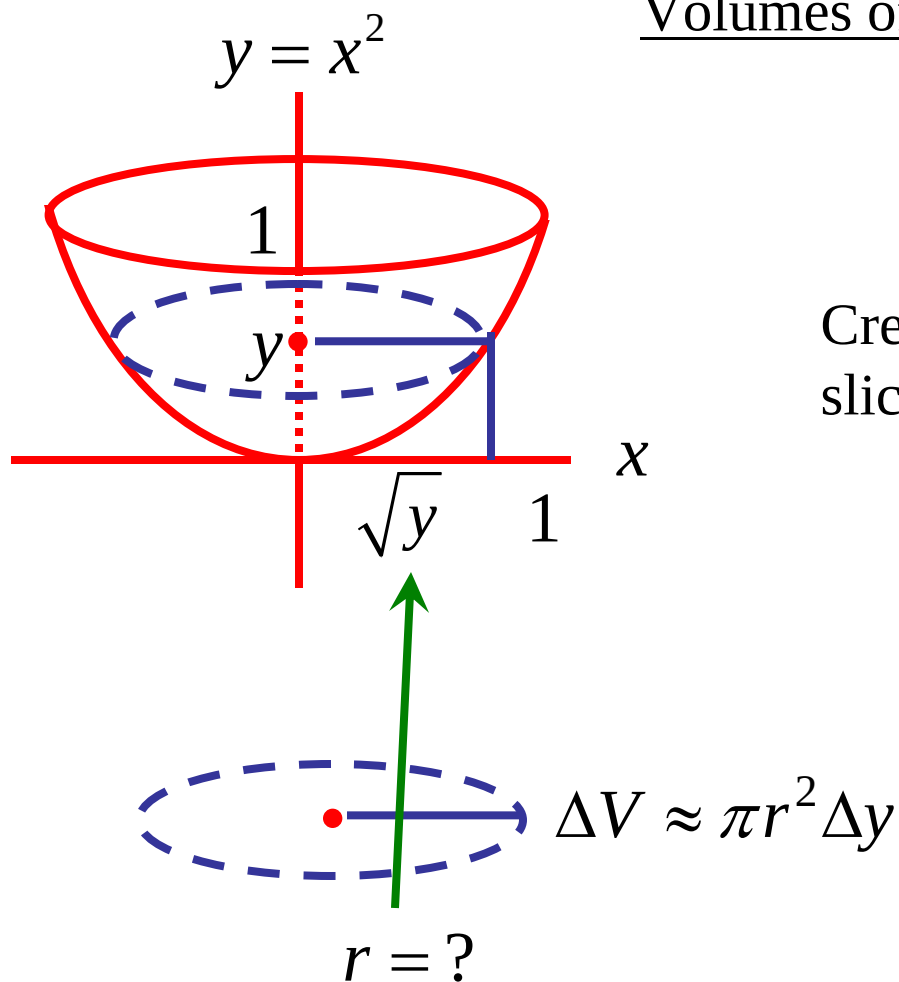
$r = ?$

Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by slicing perpendicular to the axis of rotation

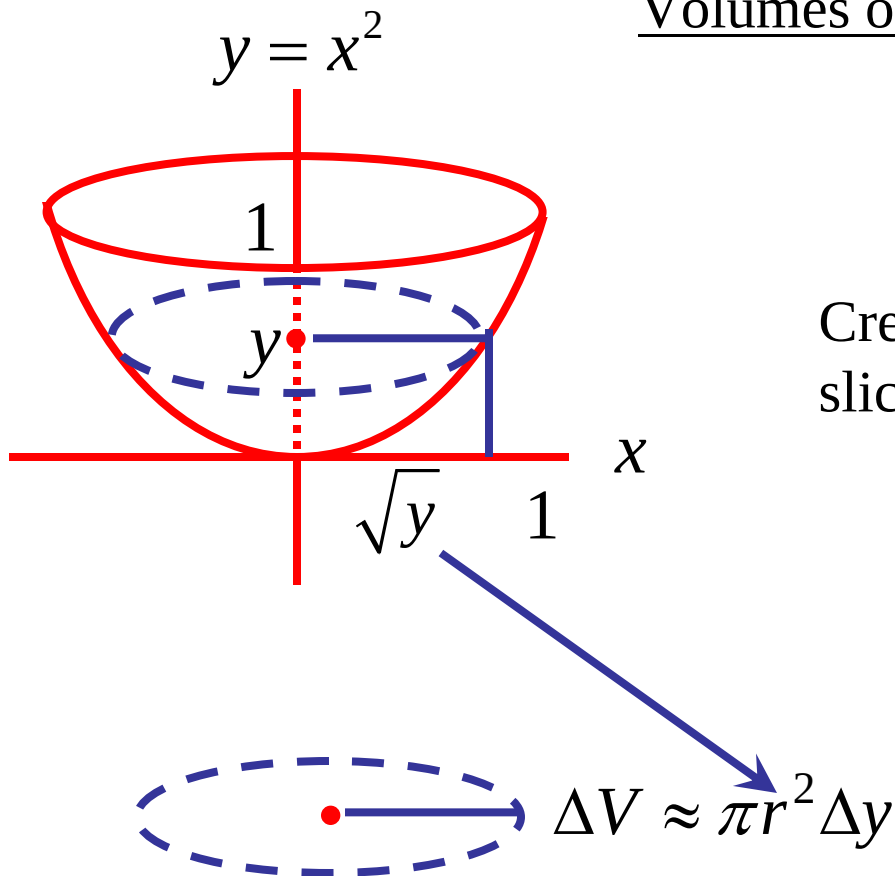


Volumes of Solids

What's the volume V ?

Slicing Method

Create approximating cylinders by slicing perpendicular to the axis of rotation



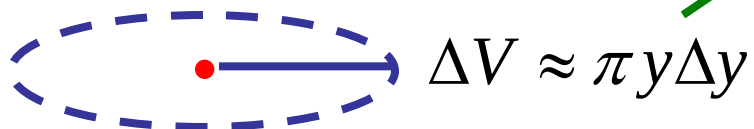
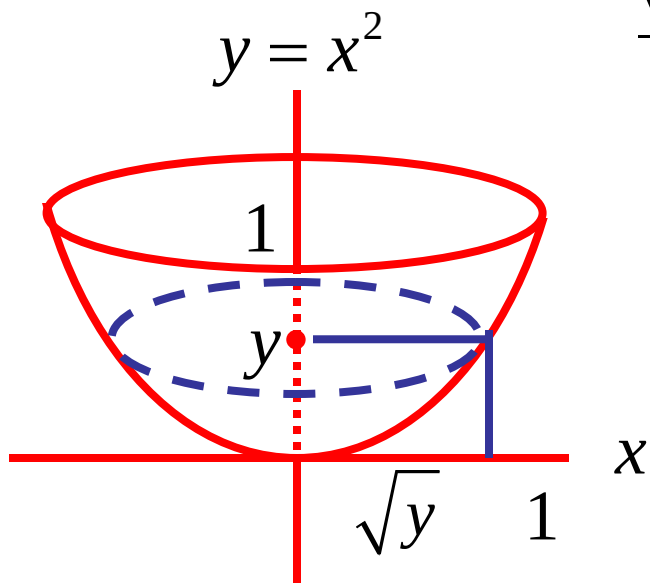
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

sum up all slices



Volumes of Solids

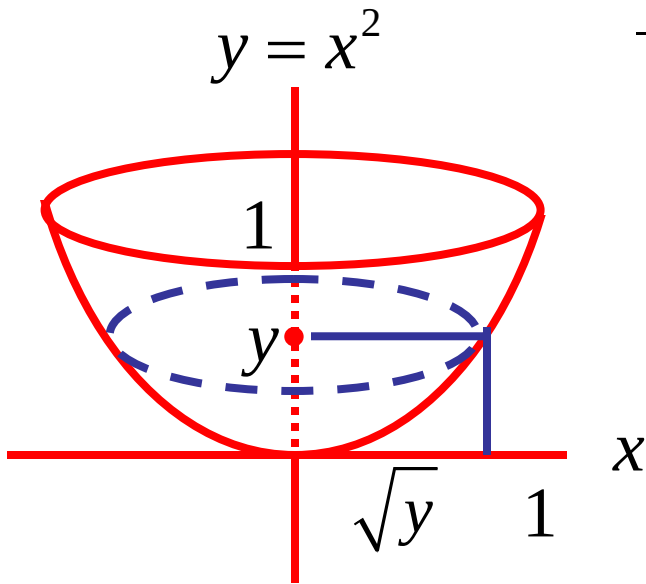
What's the volume V ?

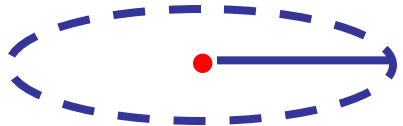
Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

Pass to the limit:

- number of slices tends to infinity
- thicknesses tend to 0





A diagram of a single slice, represented by a dashed blue ellipse. A red dot marks the center. A horizontal blue arrow points from the center to the right edge, representing the radius y . To the right of the arrow is the equation $\Delta V \approx \pi y \Delta y$.

$$\Delta V \approx \pi y \Delta y$$

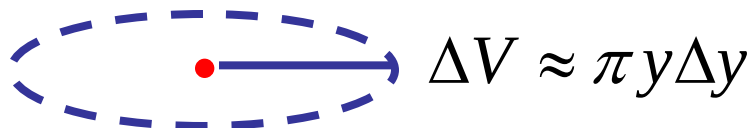
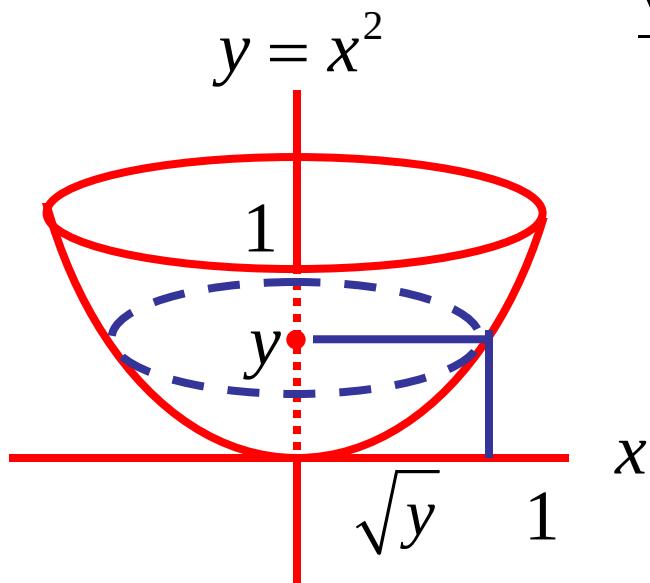
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

$$V = \int_0^1 \pi y \, dy$$



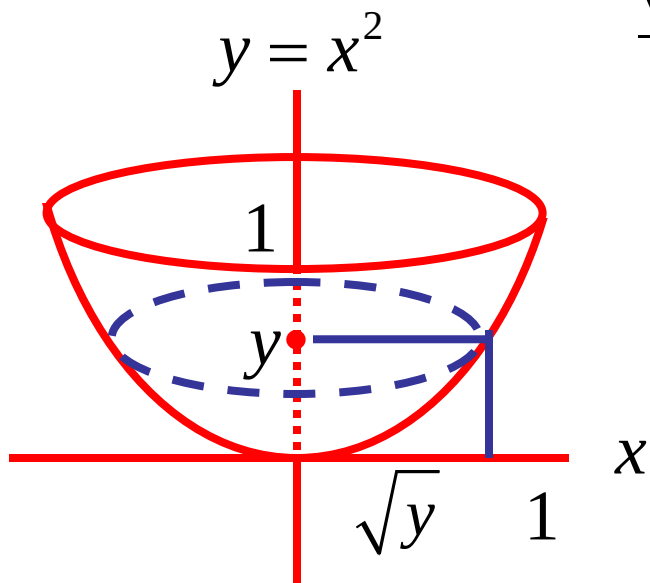
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

$$V = \int_0^1 \pi y \, dy$$



A diagram of a single slice of the solid, represented by a dashed blue ellipse. A red dot is at the center, and a blue arrow points from the center to the right edge, representing the radius. To the right of the ellipse is the equation $\Delta V \approx \pi y \Delta y$.

$$\Delta V \approx \pi y \Delta y$$

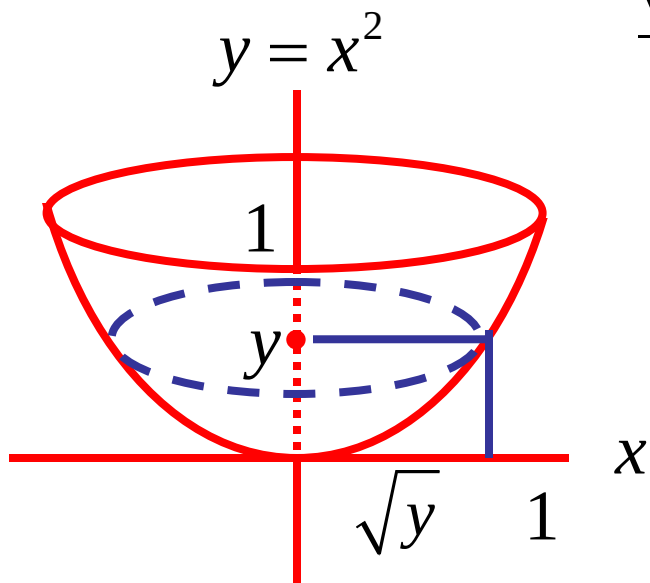
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

$$V = \int_0^1 \pi y \, dy$$



A diagram of a differential volume element ΔV , represented as a dashed blue ellipse. A red dot is at its center, and a horizontal blue arrow points from the center to the right edge, representing the radius $\sqrt{y}$. The equation $\Delta V \approx \pi y \Delta y$ is written to the right of the ellipse.

$$\Delta V \approx \pi y \Delta y$$

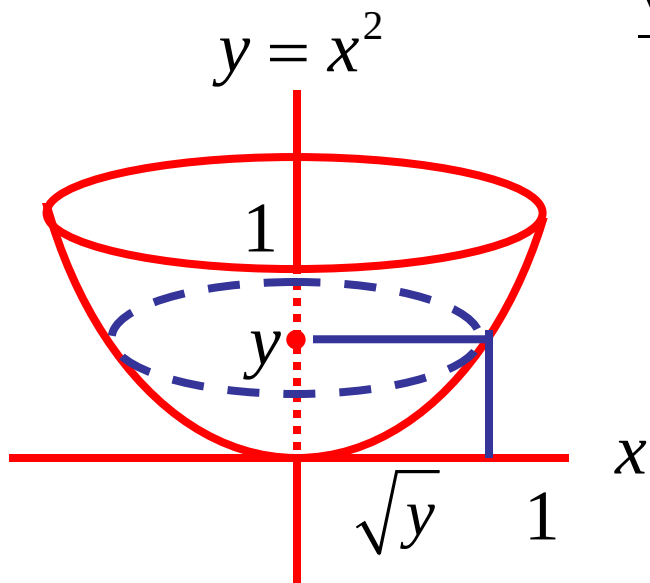
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

$$V = \int_0^1 \pi y \, dy$$



A diagram of a single slice $\Delta V \approx \pi y \Delta y$. It is represented by a dashed blue ellipse with a red dot at its center. A blue arrow points from the center of the ellipse to the right, indicating the radius.

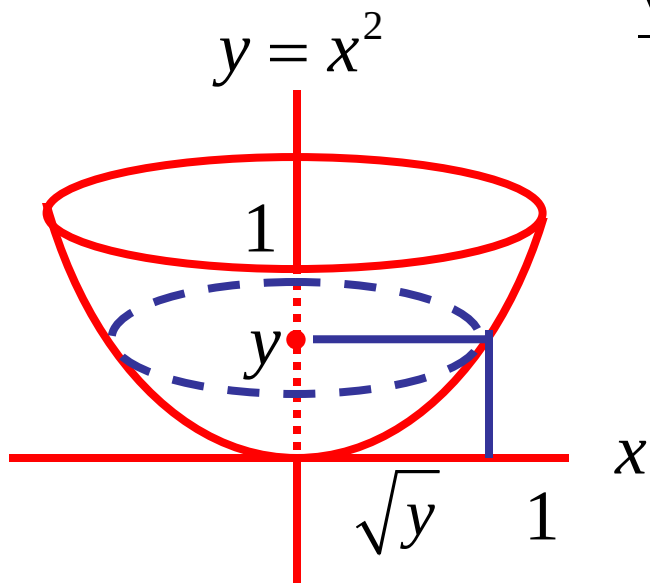
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

$$V = \int_0^1 \pi y \, dy = \frac{\pi}{2} y^2 \Big|_0^1$$



A diagram of a differential volume element ΔV . It is represented by a dashed blue ellipse with a red dot at its center. A horizontal blue arrow points from the red dot to the right, representing the radius of the slice. The equation $\Delta V \approx \pi y \Delta y$ is written to the right of the ellipse.

$$\Delta V \approx \pi y \Delta y$$

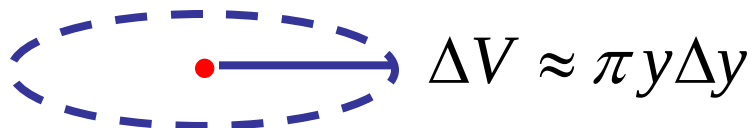
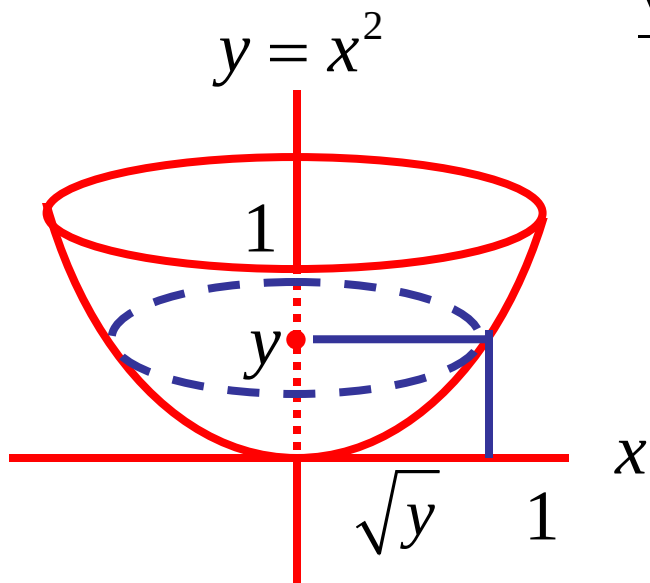
Volumes of Solids

What's the volume V ?

Slicing Method

$$V \approx \sum \Delta V \approx \sum_{y=0}^{y=1} \pi y \Delta y$$

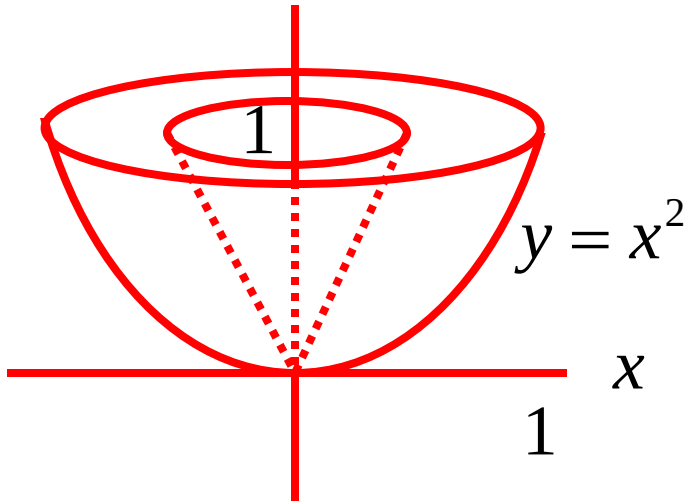
$$V = \int_0^1 \pi y \, dy = \frac{\pi}{2} y^2 \Big|_0^1 = \frac{\pi}{2}$$



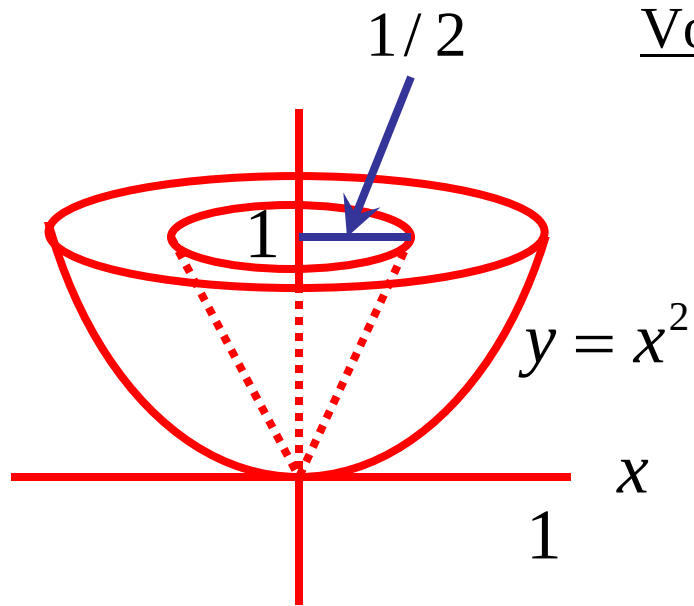
Volumes of Solids

Drill out a cone.

What's the remaining volume V ?



Volumes of Solids



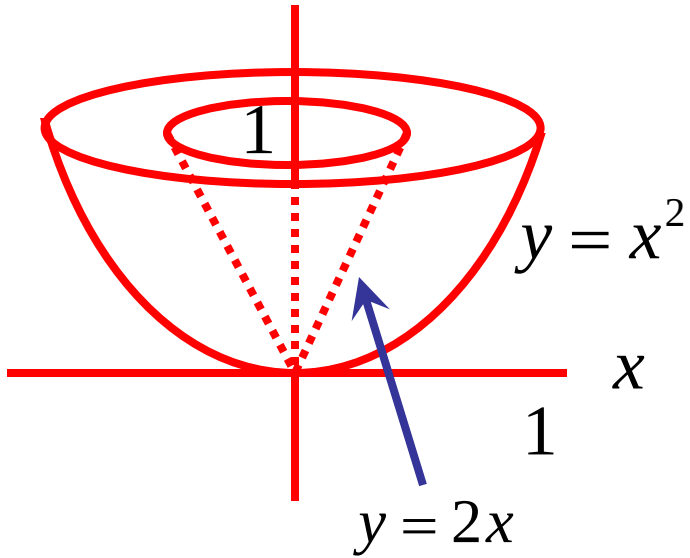
Drill out a cone.

What's the remaining volume V ?

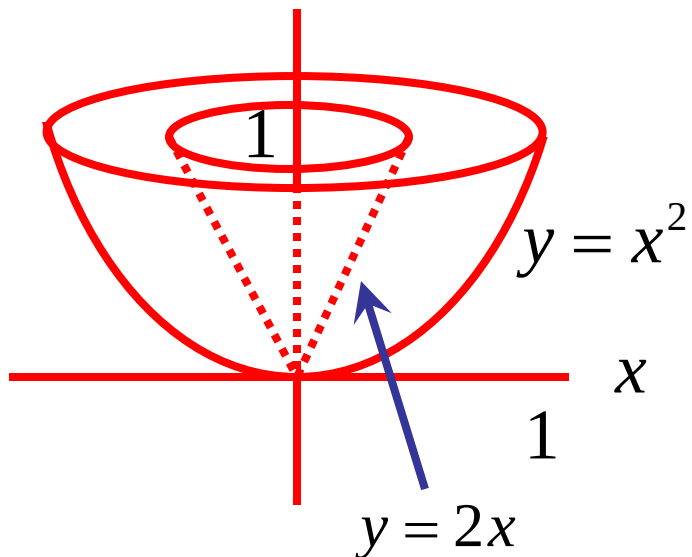
Volumes of Solids

Drill out a cone.

What's the remaining volume V ?



Volumes of Solids



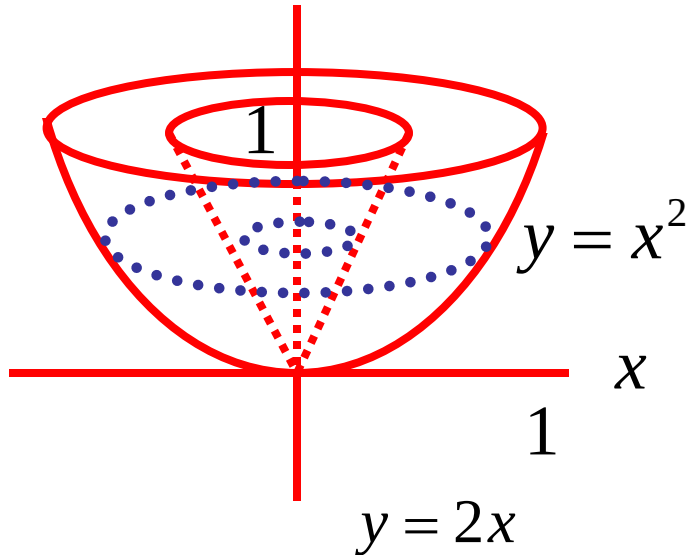
Drill out a cone.

What's the remaining volume V ?

Slicing Method

Slice perpendicular to the axis of rotation.

Volumes of Solids



Drill out a cone.

What's the remaining volume V ?

Slicing Method

Slice perpendicular to the axis of rotation.

Creates a punctured disk or washer shape.

Volumes of Solids

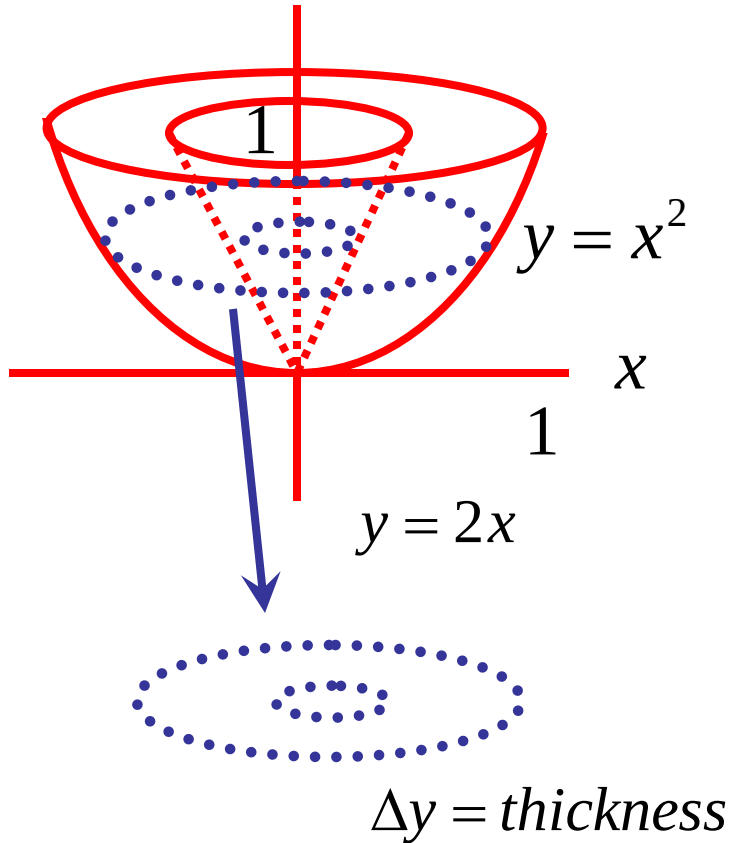
Drill out a cone.

What's the remaining volume V ?

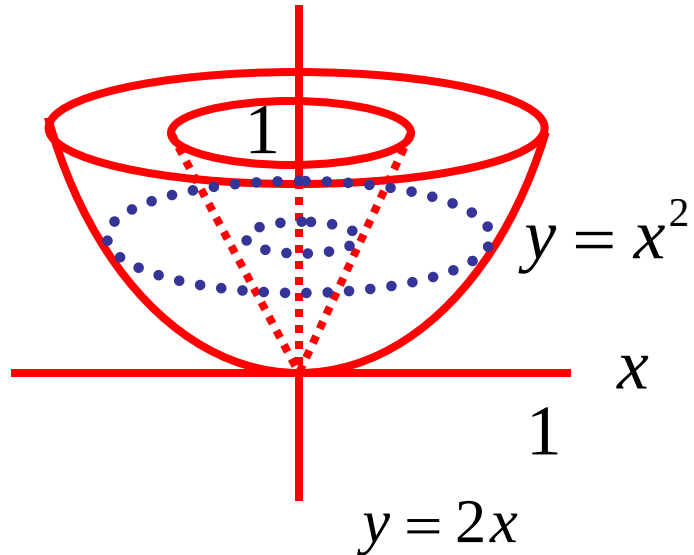
Slicing Method

Slice perpendicular to the axis of rotation.

Creates a punctured disk or washer shape.



Volumes of Solids



$\Delta y = \text{thickness}$

Drill out a cone.

What's the remaining volume V ?

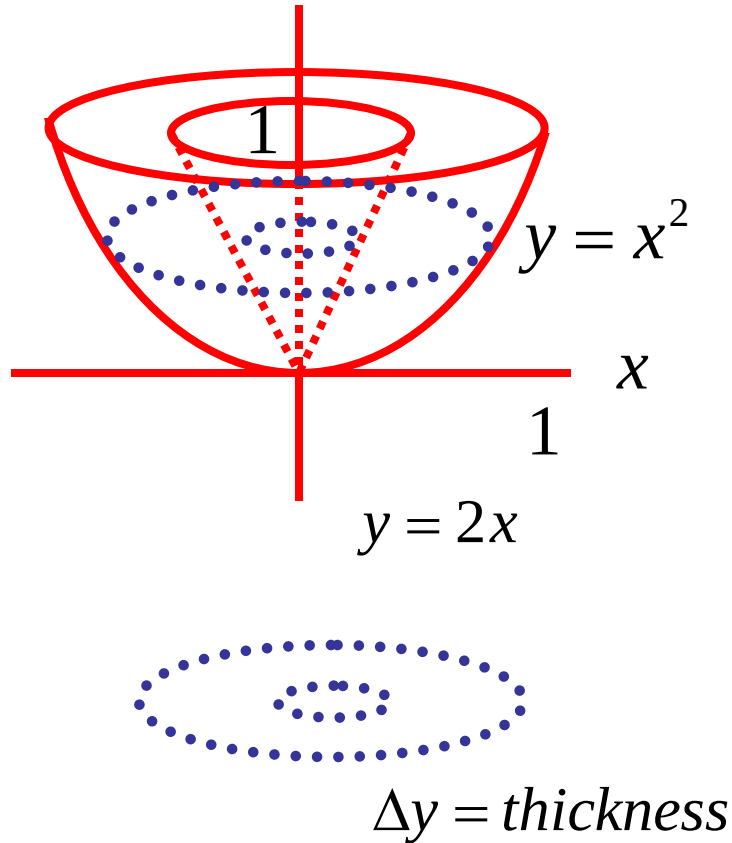
Slicing Method

Slice perpendicular to the axis of rotation.

Creates a punctured disk or washer shape.

$$\Delta V = \text{outside volume} - \text{inside volume}$$

Volumes of Solids



Drill out a cone.

What's the remaining volume V ?

Slicing Method

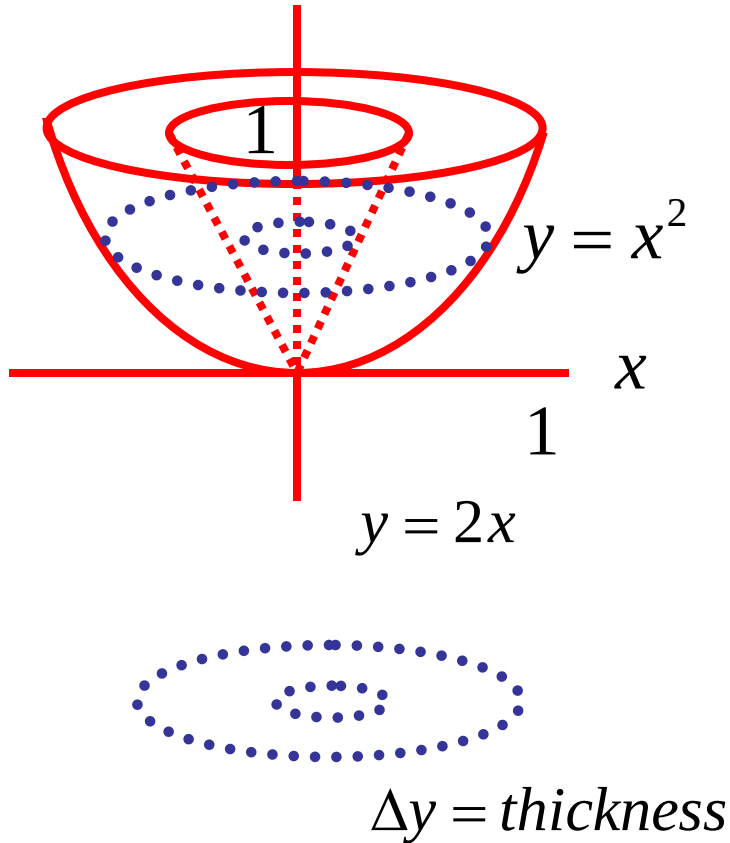
Slice perpendicular to the axis of rotation.

Creates a punctured disk or washer shape.

$$\Delta V = \text{outside volume} - \text{inside volume}$$

$$\approx \pi r_{\text{out}}^2 \Delta y - \pi r_{\text{in}}^2 \Delta y$$

Volumes of Solids



Drill out a cone.

What's the remaining volume V ?

Slicing Method

Slice perpendicular to the axis of rotation.

Creates a punctured disk or washer shape.

$$\Delta V = \text{outside volume} - \text{inside volume}$$

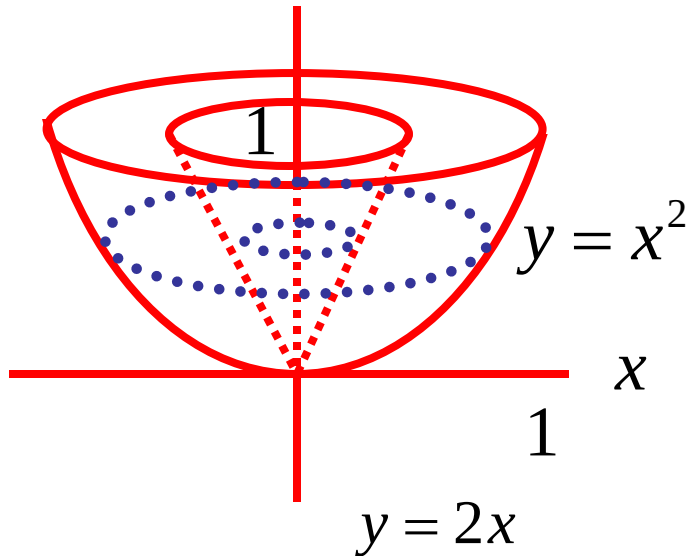
$$\approx \pi r_{out}^2 \Delta y - \pi r_{in}^2 \Delta y$$

$$\approx \pi \left(\sqrt{y} \right)^2 \Delta y - \pi \left(\frac{y}{2} \right)^2 \Delta y$$

Volumes of Solids

Drill out a cone.

What's the remaining volume V ?

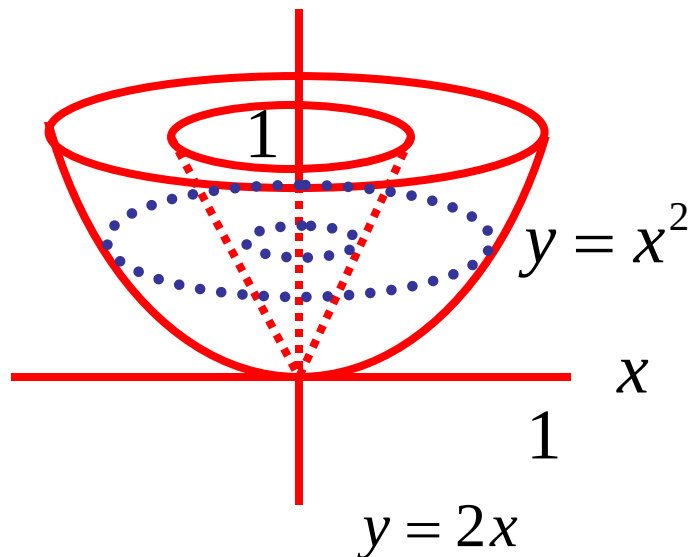


$$V \approx \sum \Delta V$$



$$\Delta y = \text{thickness}$$

Volumes of Solids



$\Delta y = \text{thickness}$

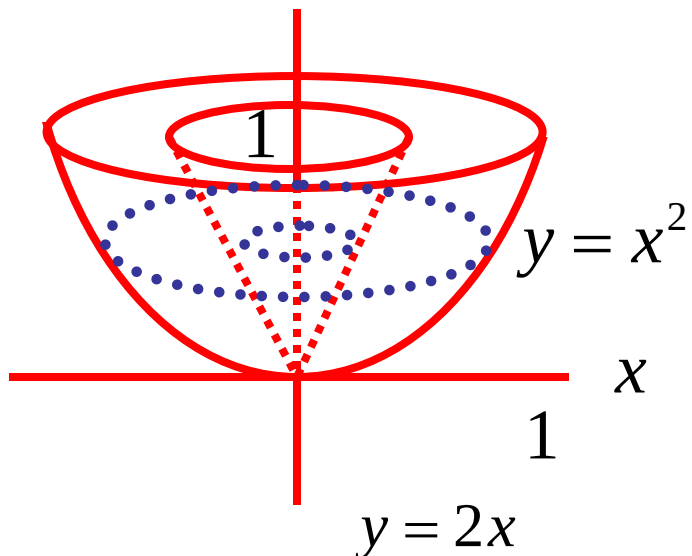
Drill out a cone.

What's the remaining volume V ?

$$V \approx \sum \Delta V$$

$$\approx \sum_{y=0}^{y=1} \left(\pi (\sqrt{y})^2 \Delta y - \pi \left(\frac{y}{2} \right)^2 \Delta y \right)$$

Volumes of Solids



$\Delta y = \text{thickness}$

Drill out a cone.

What's the remaining volume V ?

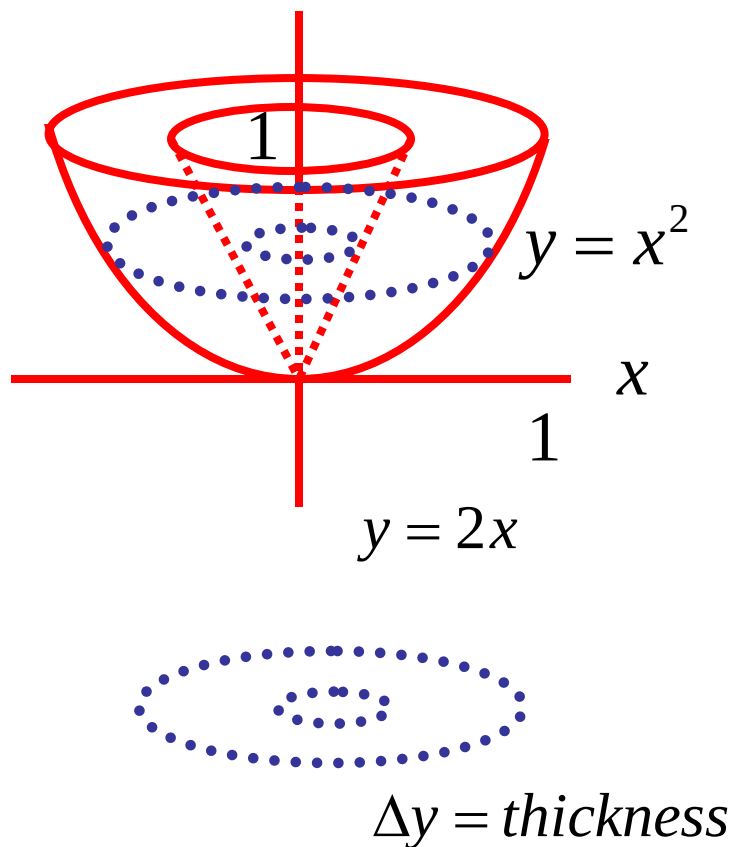
$$V \approx \sum \Delta V$$

$$\approx \sum_{y=0}^{y=1} \left(\pi (\sqrt{y})^2 - \pi \left(\frac{y}{2} \right)^2 \right) \Delta y$$

Pass to the limit:

- number of slices tends to infinity
- thicknesses tend to 0

Volumes of Solids



Drill out a cone.

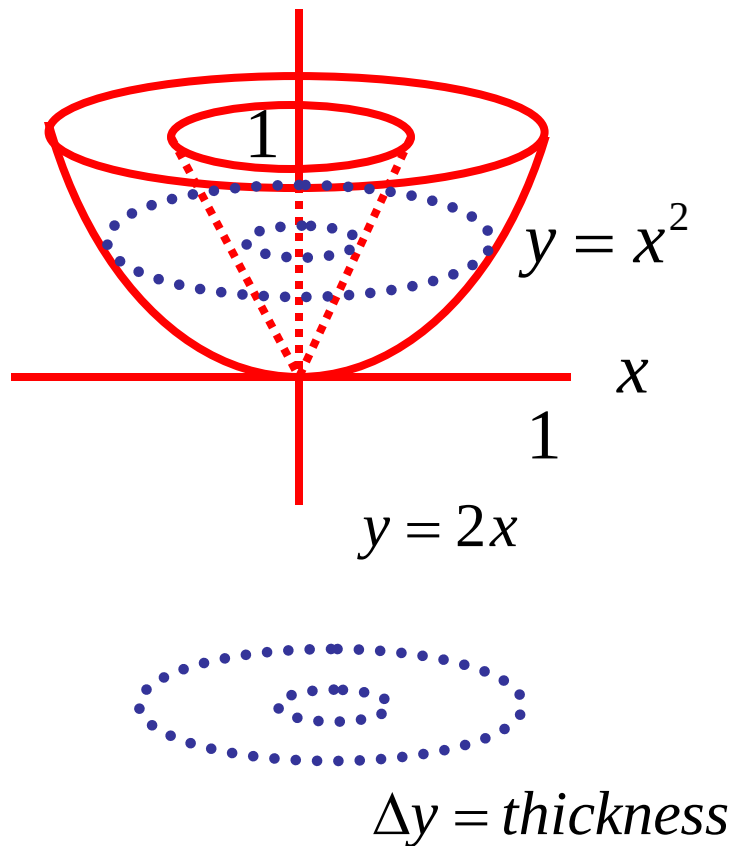
What's the remaining volume V ?

$$\begin{aligned} V &\approx \sum \Delta V \\ &\approx \sum_{y=0}^{y=1} \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) \Delta y \\ &\quad \downarrow \\ V &= \int_0^1 \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) dy \end{aligned}$$

Volumes of Solids

Drill out a cone.

What's the remaining volume V ?

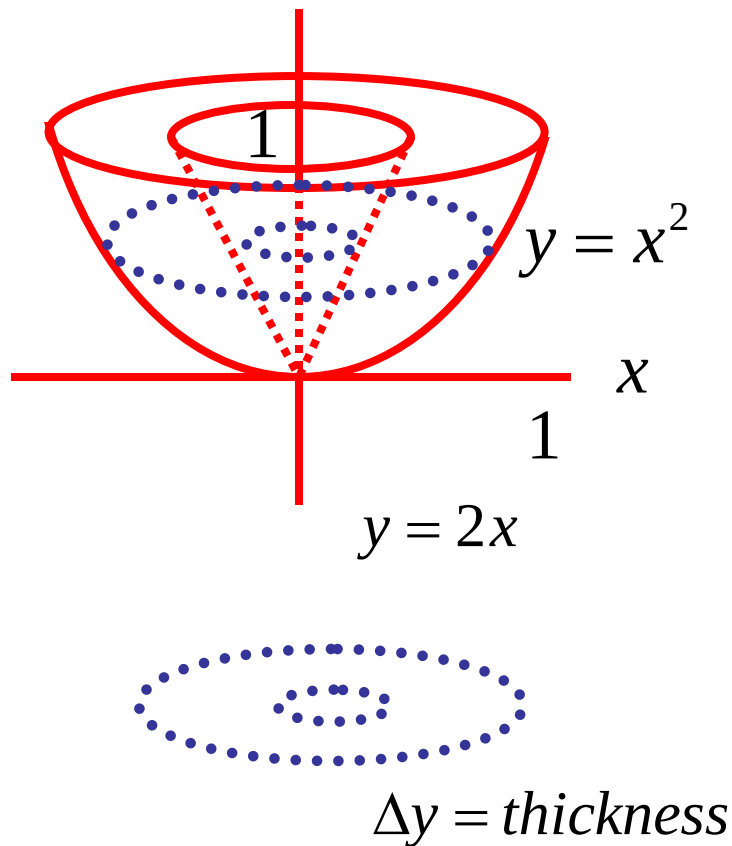


$$V \approx \sum \Delta V$$

$$\approx \sum_{y=0}^{y=1} \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) \Delta y$$

$$V = \int_0^1 \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) dy$$

Volumes of Solids



Drill out a cone.

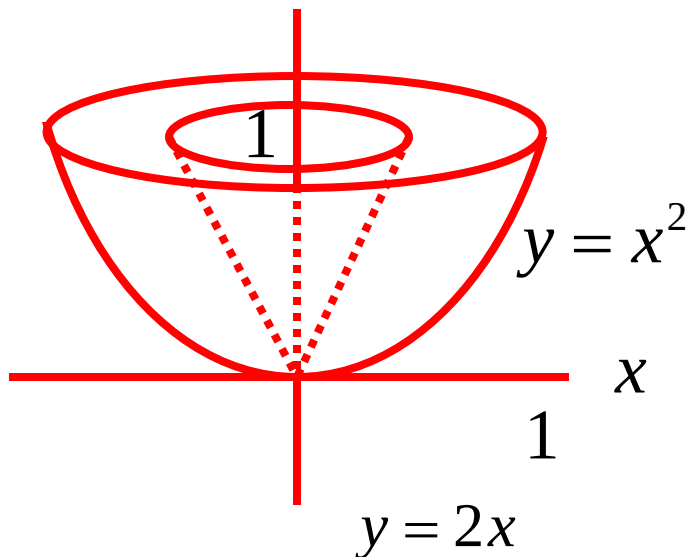
What's the remaining volume V ?

$$V \approx \sum \Delta V$$

$$\approx \sum_{y=0}^{y=1} \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) \Delta y$$

$$V = \int_0^1 \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) dy$$

Volumes of Solids

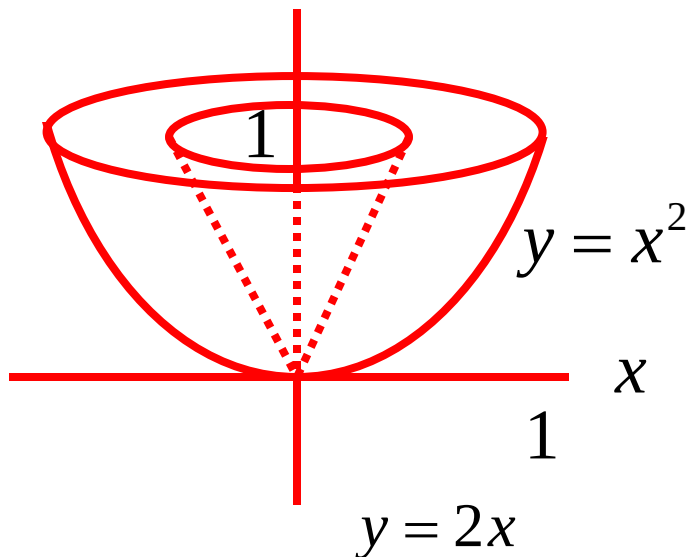


Drill out a cone.

What's the remaining volume V ?

$$V = \int_0^1 \left(\pi \left(\sqrt{y} \right)^2 - \pi \left(\frac{y}{2} \right)^2 \right) dy$$

Volumes of Solids

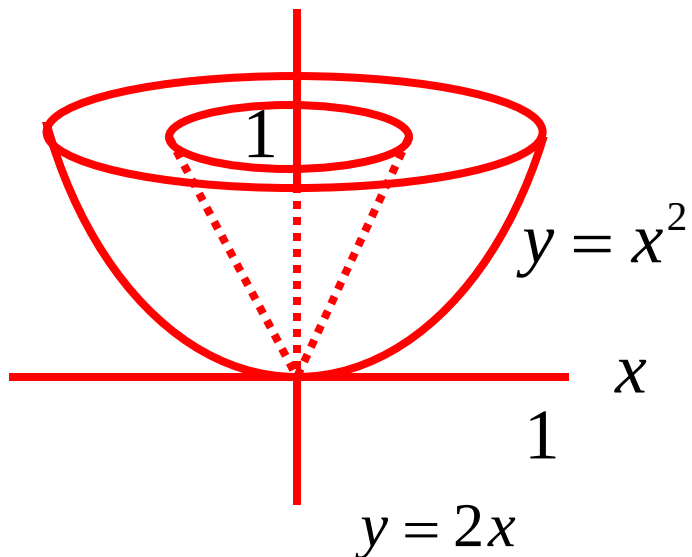


Drill out a cone.

What's the remaining volume V ?

$$V = \pi \int_0^1 \left(y - \frac{1}{4} y^2 \right) dy$$

Volumes of Solids

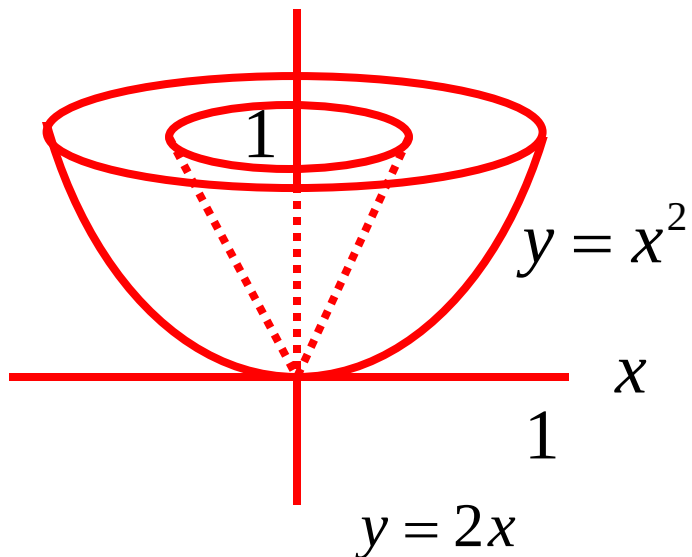


Drill out a cone.

What's the remaining volume V ?

$$\begin{aligned} V &= \pi \int_0^1 \left(y - \frac{1}{4} y^2 \right) dy \\ &= \pi \left(\frac{1}{2} y^2 - \frac{1}{12} y^3 \right) \Big|_0^1 \end{aligned}$$

Volumes of Solids

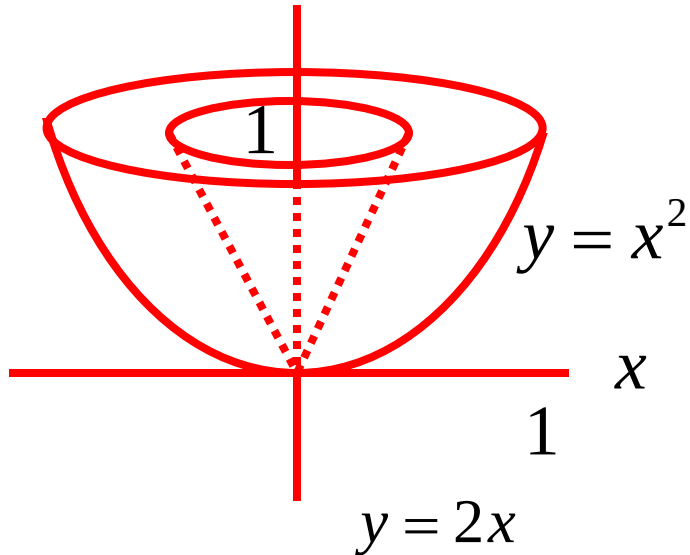


Drill out a cone.

What's the remaining volume V ?

$$\begin{aligned} V &= \pi \int_0^1 \left(y - \frac{1}{4} y^2 \right) dy \\ &= \pi \left(\frac{1}{2} y^2 - \frac{1}{12} y^3 \right) \bigg|_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{12} \right) = \frac{5}{12} \pi \end{aligned}$$

Volumes of Solids



Drill out a cone.

What's the remaining volume V ?

$$\begin{aligned} V &= \pi \int_0^1 \left(y - \frac{1}{4} y^2 \right) dy \\ &= \pi \left(\frac{1}{2} y^2 - \frac{1}{12} y^3 \right) \bigg|_0^1 \\ &= \pi \left(\frac{1}{2} - \frac{1}{12} \right) = \frac{5}{12} \pi \end{aligned}$$

Note: If we recall the formula for the volume of a cone then, having calculated the volume of the parabolic bowl in the previous example, we could get this answer by subtraction.

Still ... it was good practice to calculate the answer as a solid of revolution!