

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, October 2

<http://math.arizona.edu/~cushing/250a.html>

Chapter 7: Integration

Sections 1 – 6

Two main themes:

- ✓ ➤ Calculation of anti-derivatives (integrals)
One goal is to evaluate integrals using
the *Fundamental Theorem of Calculus*
- Numerical approximation of integrals

Chapter 7: Integration

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- ✓ ➤ Calculation of anti-derivatives (integrals)
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An anti-derivative is not available.

$$\int_0^1 e^{-x^2} dx$$

Chapter 7: Integration

Sections 1 – 6

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- ✓ ➤ Calculation of anti-derivatives (integrals)
One goal is to evaluate integrals using
the *Fundamental Theorem of Calculus*
- Numerical approximation of integrals

An anti-derivative is not available.

A formula for the integrand is not available.

5. Numerical approximation of integrals

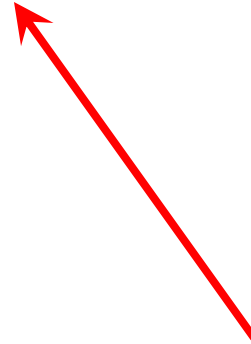
The accumulated amount of a chemical (drug) over time is important in toxicity studies.

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OK if the amount remains constant
over the time interval

5. Numerical approximation of integrals

The accumulated amount of a chemical (drug) over time is important in toxicity studies.

Cumulative effect (dose) of a drug = amount $\times$ length of time present.

Otherwise we integrate amount of drug $A(t)$ over total time of interest.

$$\text{Cumulative dose} = \int_a^b A(t) dt$$

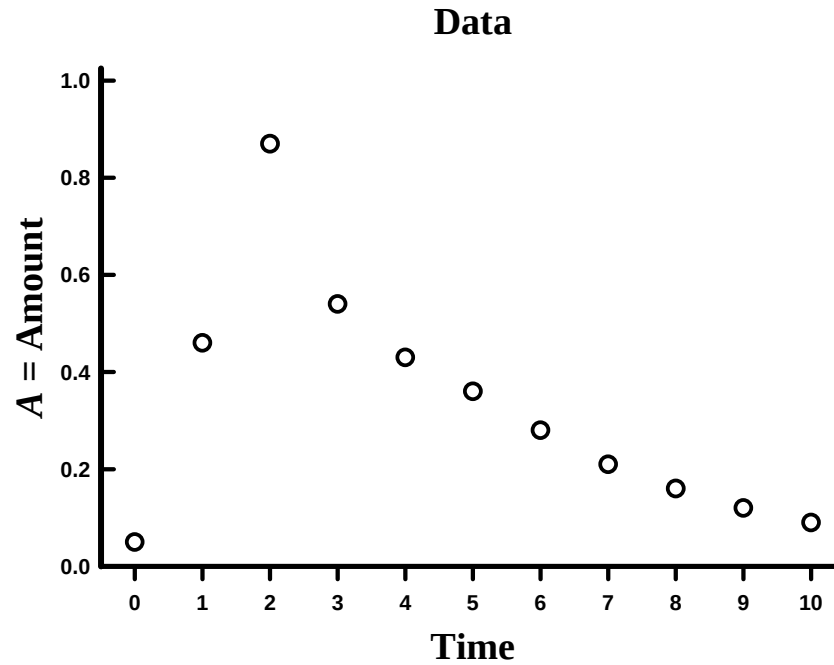
5. Numerical approximation of integrals

Estimate the cumulative dose from the data below from hour a to b .

$$\int_a^b A(t) dt \approx ?$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

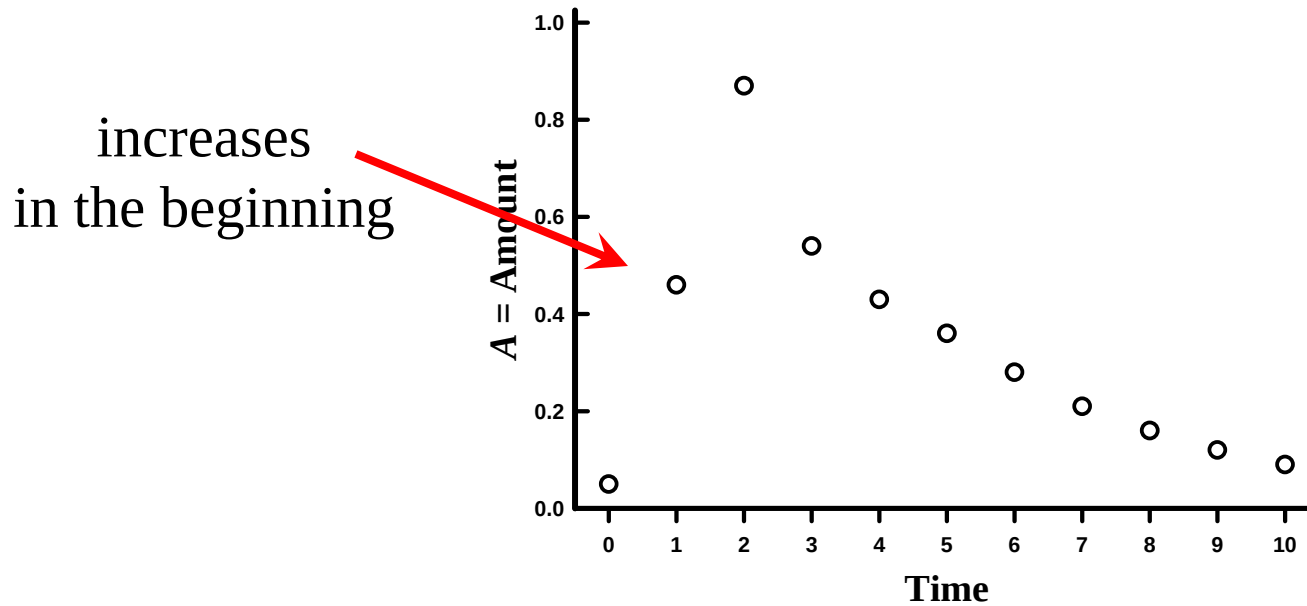
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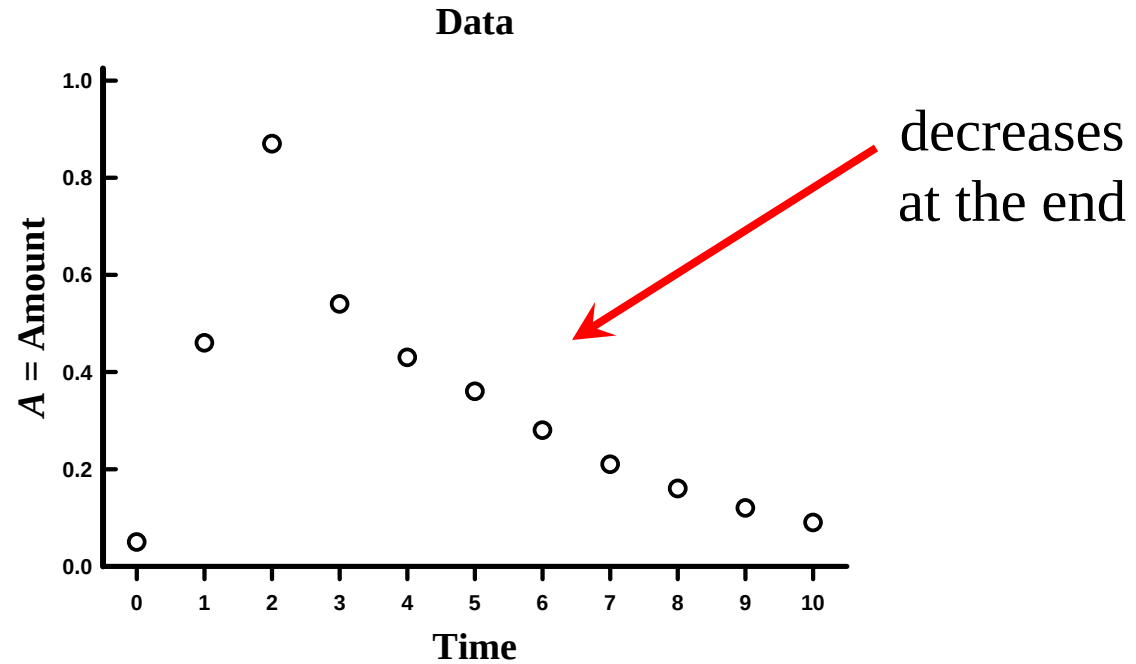
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Data



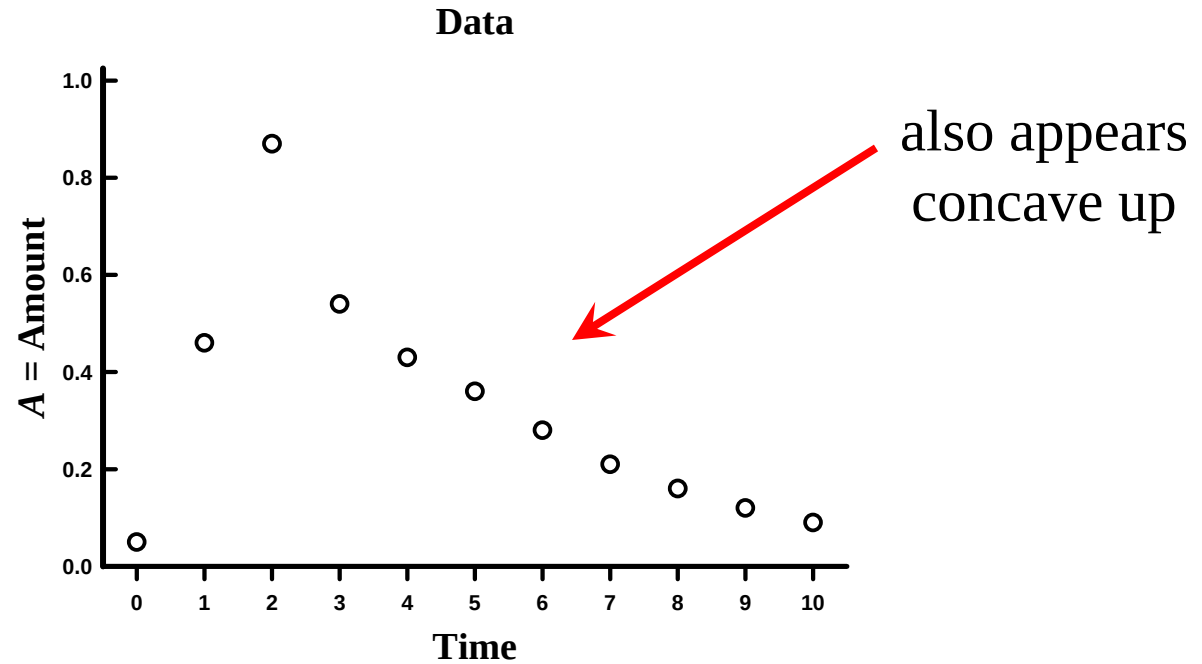
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5. Numerical approximation of integrals

Definition of the definite integral (text, page 252)

$$\int_a^b f(t) dt = \lim_{n \rightarrow +\infty} \sum_{i=1}^n f(c_i) \Delta t_i$$

$$a = t_0 < t_1 < \cdots < t_n = b, \quad \Delta t_i = t_i - t_{i-1}, \quad t_{i-1} < c_i < t_i$$

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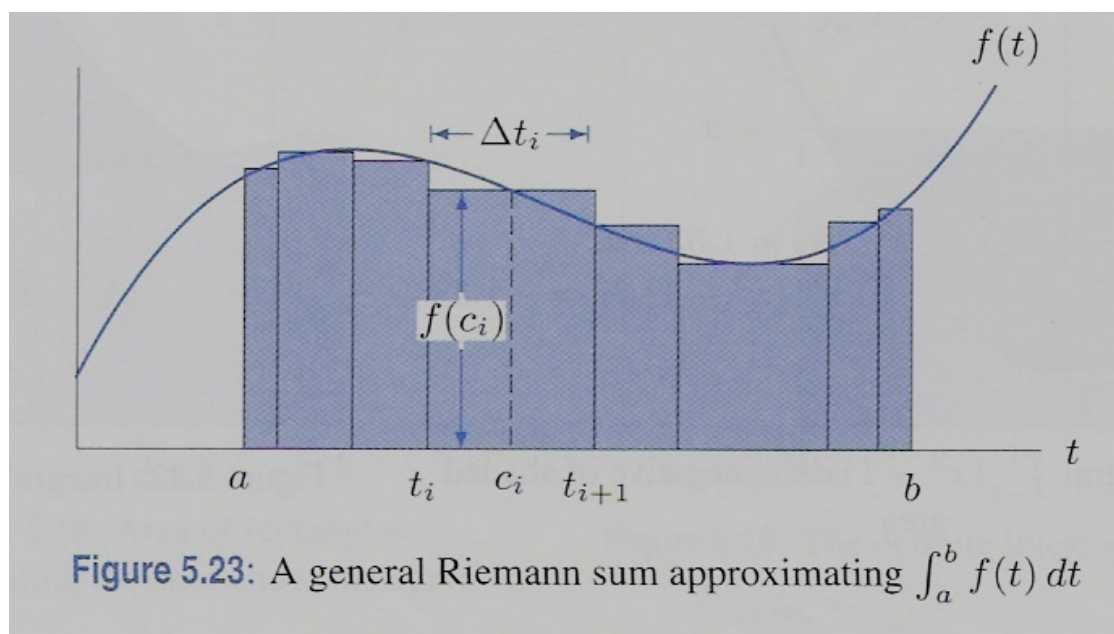


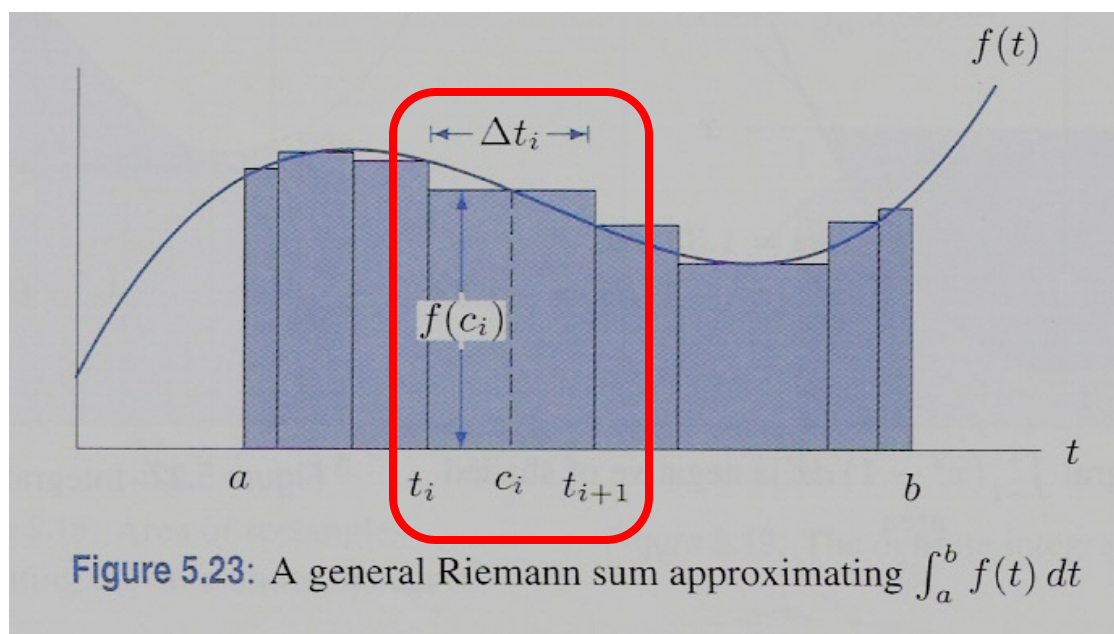
Figure 5.23: A general Riemann sum approximating $\int_a^b f(t) dt$

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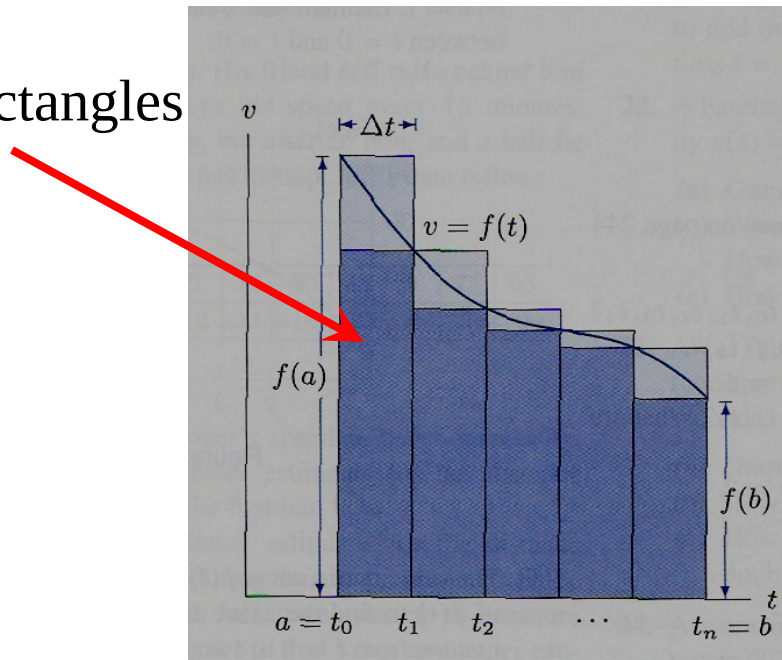
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$$\int_a^b f(t) dt = \lim_{n \rightarrow +\infty} (\text{Right-hand sum}) = \lim_{n \rightarrow +\infty} \sum_{i=1}^n f(t_i) \Delta t$$

dark blue rectangles

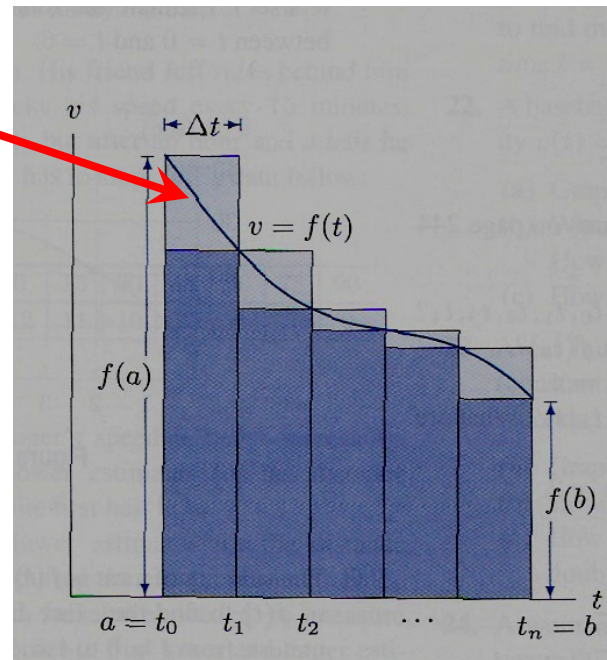


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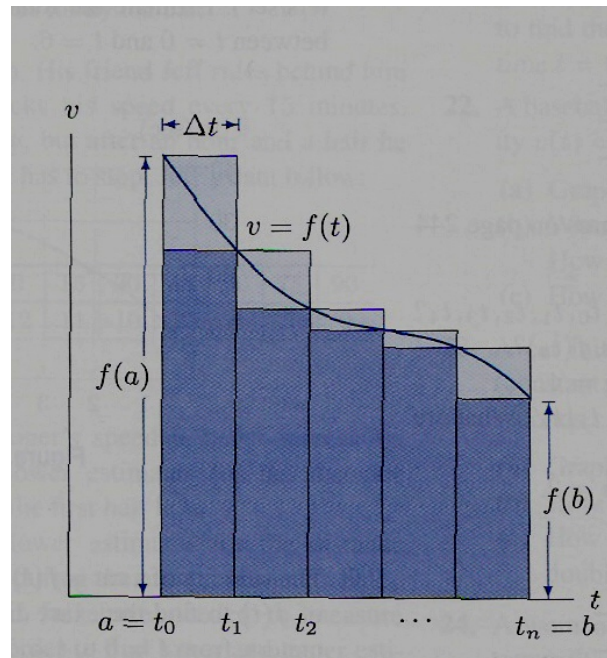
light blue rectangles



5. Numerical approximation of integrals

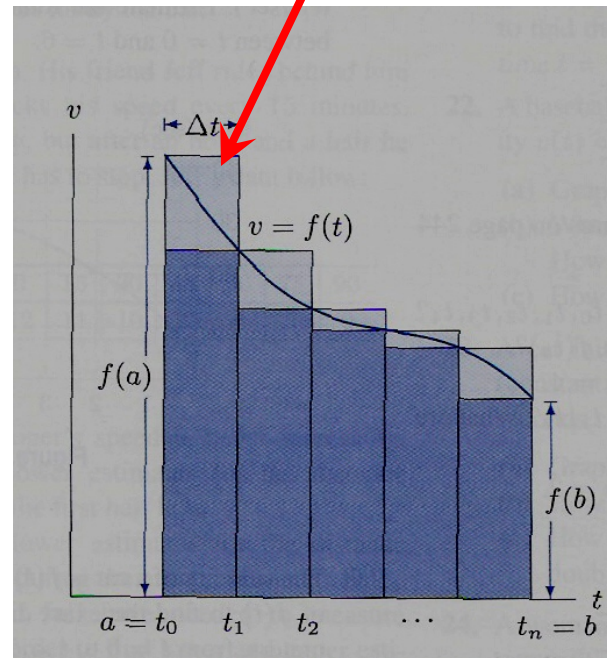
$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f(t_i) \Delta t = \text{Right-hand sum} = \text{RIGHT}(n)$$

$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f(t_i) \Delta t = \text{Left-hand sum} = \text{LEFT}(n)$$



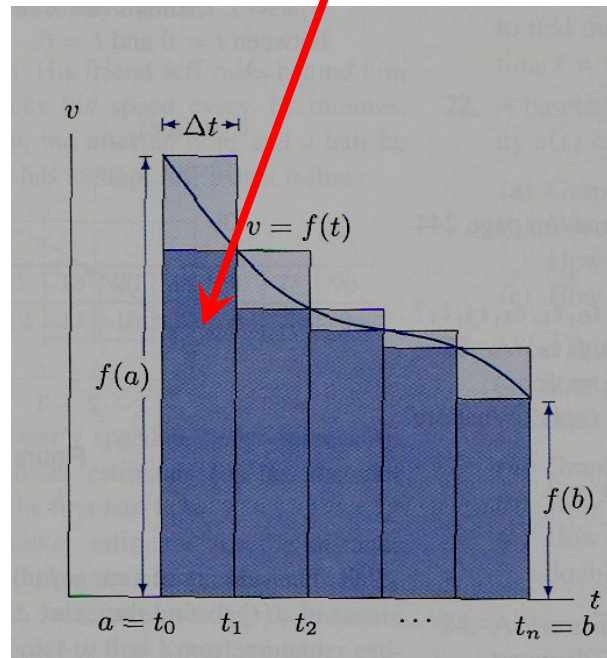
5. Numerical approximation of integrals

In this figure, we see that the *left* rectangles *over estimate* because $f(t)$ is *decreasing*.



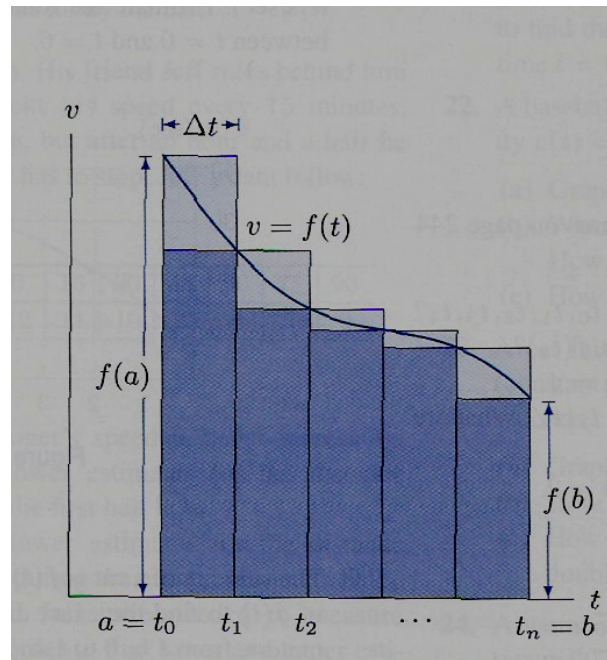
5. Numerical approximation of integrals

In this figure, we see that the *right* rectangles *under estimate* because $f(t)$ is *decreasing*.



5. Numerical approximation of integrals

If $f(t)$ is *decreasing* then: $\text{RIGHT}(n) \leq \int_a^b f(t) dt \leq \text{LEFT}(n)$



5. Numerical approximation of integrals

If $f(t)$ is *decreasing* then: $\text{RIGHT}(n) \leq \int_a^b f(t) dt \leq \text{LEFT}(n)$

If $f(t)$ is *increasing* then: $\text{LEFT}(n) \leq \int_a^b f(t) dt \leq \text{RIGHT}(n)$

Error Bounds

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
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The data provides the values of $A(t)$ at endpoints of unit length subintervals.

... so we can calculate LEFT(5) and RIGHT(5) with $\Delta t = 1$.

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The data provides the values of $A(t)$ at endpoints of unit length subintervals.

... so we can calculate LEFT(5) and RIGHT(5) with $\Delta t = 1$.

Note data A taken at hours 5, 6, 7, 8, 9, 10 *decreases*.

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
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If we assume $A(t)$ decreases during the last five hours, then

$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

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$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

$$0.28+0.21+0.16+0.12+0.09 \leq \int_5^{10} A(t) dt \leq \text{LEFT}(10)$$

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Time t (hour)						5	6	7	8	9	10
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$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

$$0.86 \leq \int_5^{10} A(t) dt \leq \text{LEFT}(10)$$

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$$0.86 \leq \int_5^{10} A(t) dt \leq 0.36+0.28+0.21+0.16+0.12$$

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How might we improve these approximations?

Estimate the cumulative dose during the last 5 hours.

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Time t (hour)						5	6	7	8	9	10
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In using LEFT(5) we effectively assumed the dose
was *constant* at 0.36 from hour 5 to 6.

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In using LEFT(5) we effectively assumed the dose was *constant* at 0.36 from hour 5 to 6.

In using RIGHT(5) we effectively assumed the dose was *constant* at 0.28 from hour 5 to 6.

Almost assuredly neither assumption is true.

Estimate the cumulative dose during the last 5 hours.

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In using RIGHT(5) we effectively assumed the dose was *constant* at 0.28 from hour 5 to 6.

Almost assuredly neither assumption is true.
Most likely the dose *continuously decreased* from 0.36 to 0.28 from hour 5 to 6.

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
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For example, assume a dose of $\frac{0.36 + 0.28}{2} = 0.32$ from hour 5 to 6.

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whose height = the *average* of the endpoint heights

... i.e., the heights used in LEFT(5) and RIGHT(5).

$$\frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}$$

Trapezoid Rule

Define: $\text{TRAP}(n) = \frac{\text{LEFT}(n) + \text{RIGHT}(n)}{2}$

Trapezoid Rule

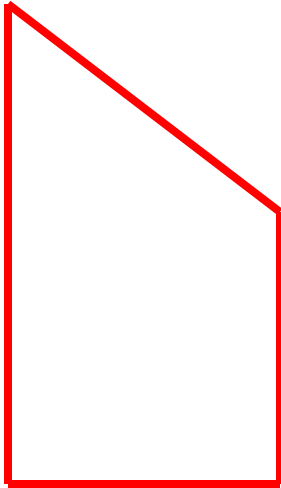
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Why is this called the Trapezoid Rule?

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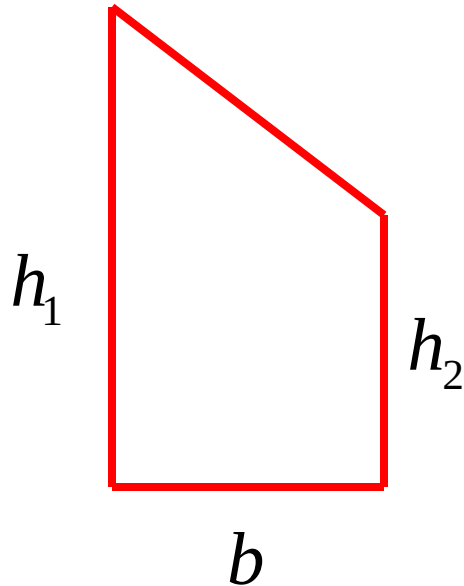


area of a trapezoid = base $\times$ *average* height

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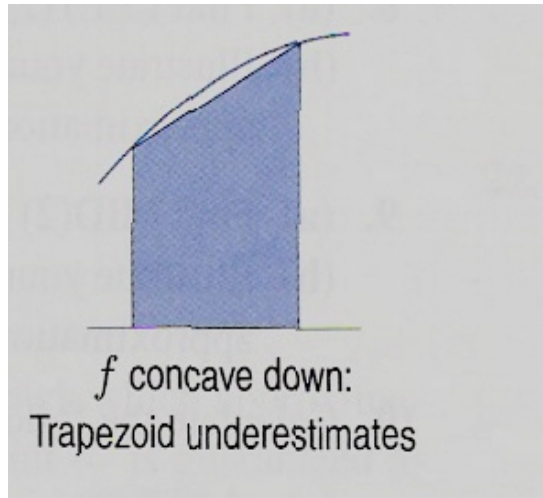


area of a trapezoid = base $\times$ *average* height

$$= b \frac{h_1 + h_2}{2}$$

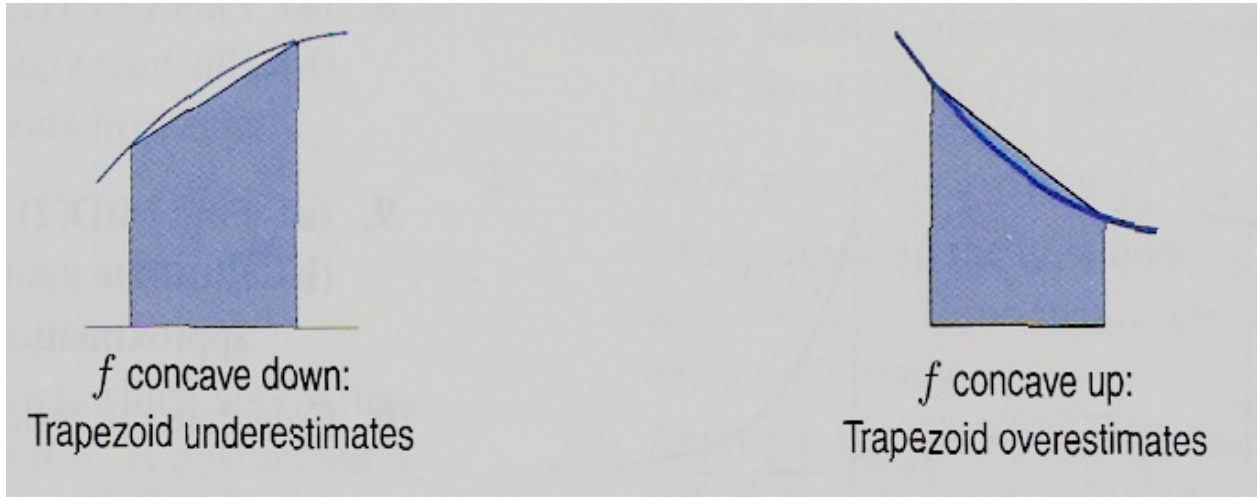
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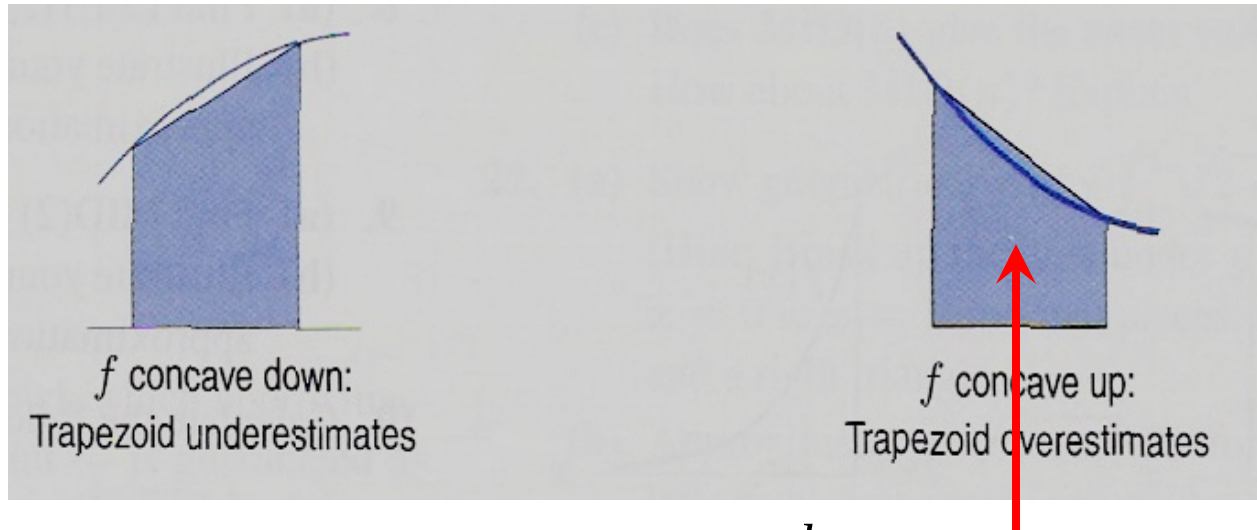
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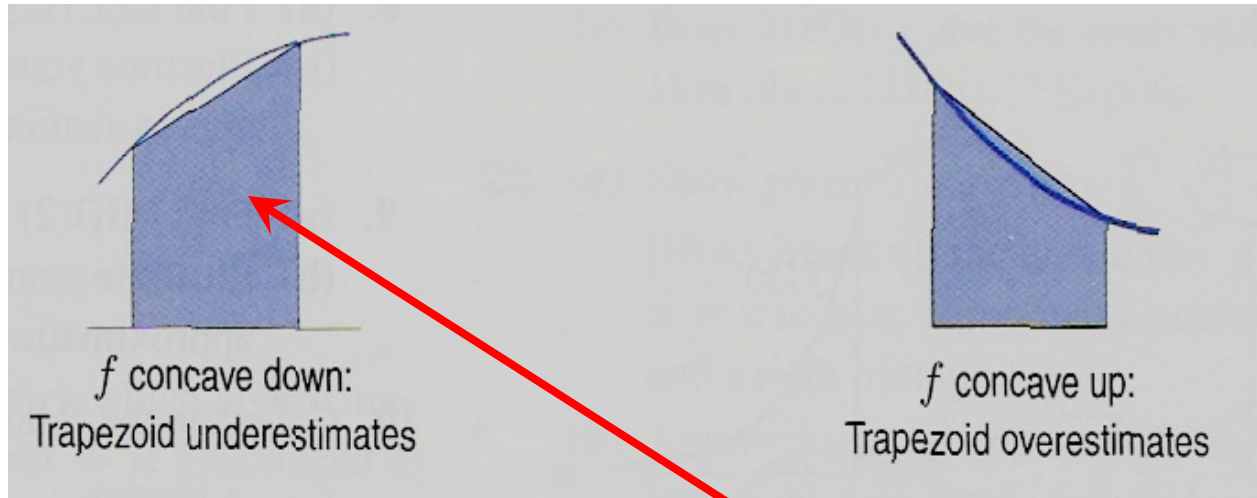
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If $f(t)$ is concave up then: $\int_a^b f(t) dt \leq \text{TRAP}(n)$

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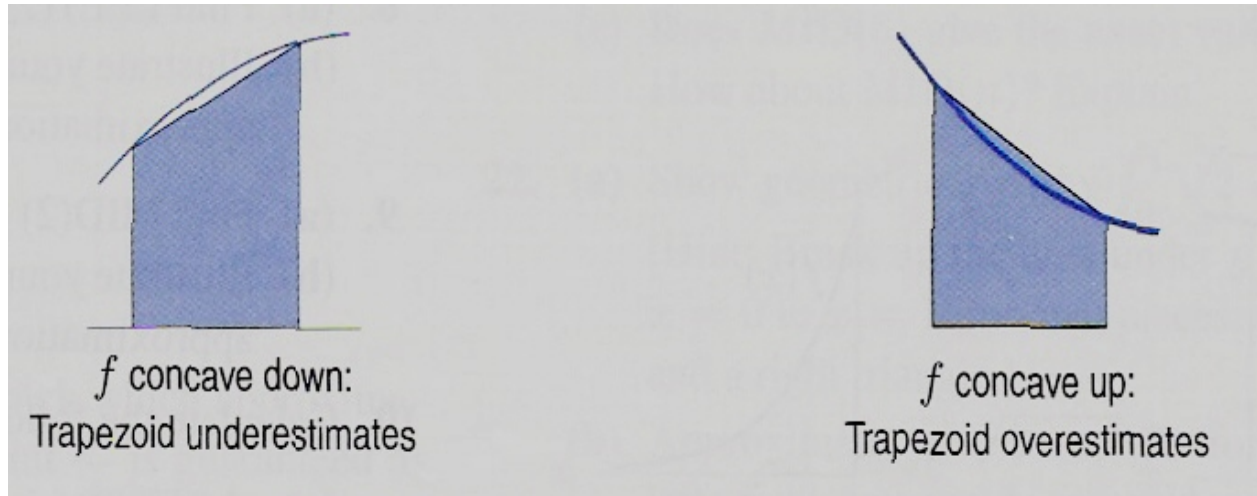


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One-sided error bounds only

Trapezoid Rule

We can get *two-sided* error bounds by using another approximation method ...one that uses averages on the *horizontal* axis (instead of the vertical axis).

Midpoint Rule

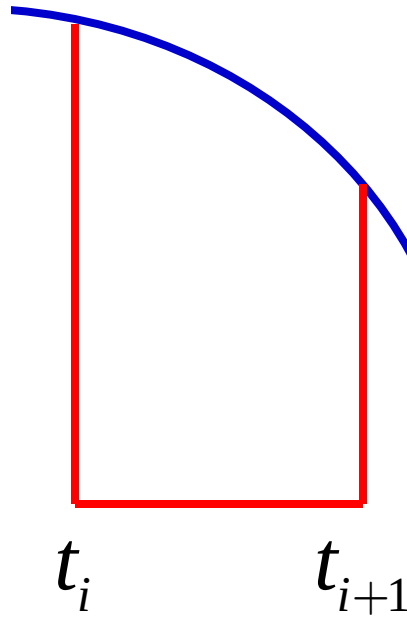
$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



midpoints
of the subintervals

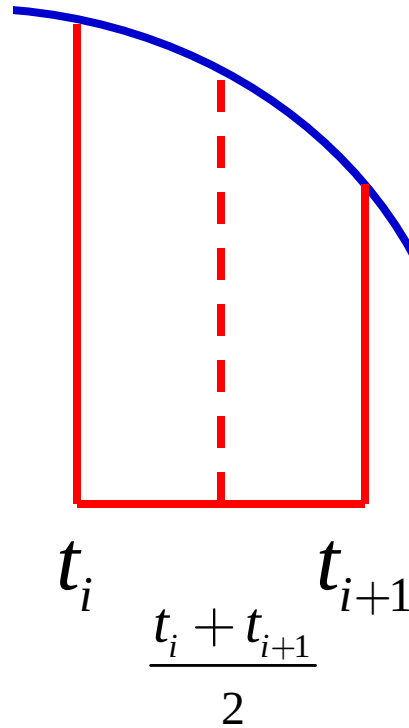
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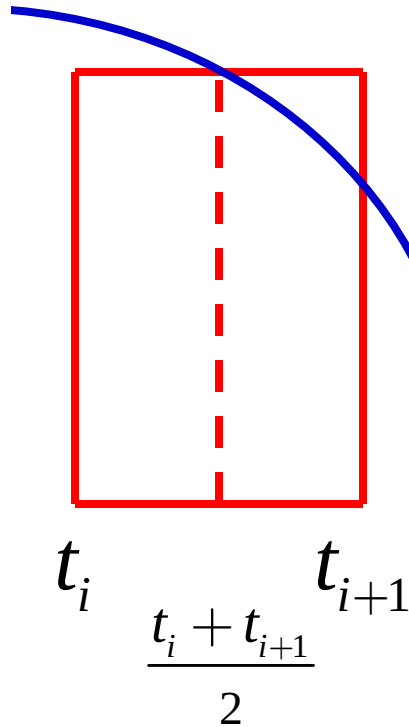
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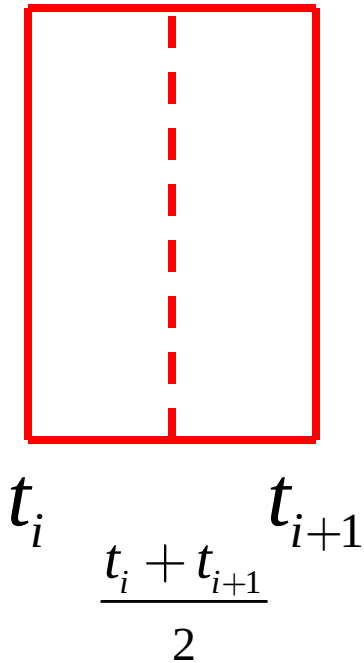
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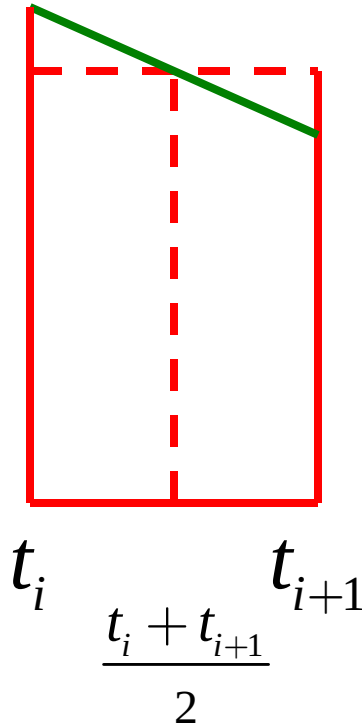
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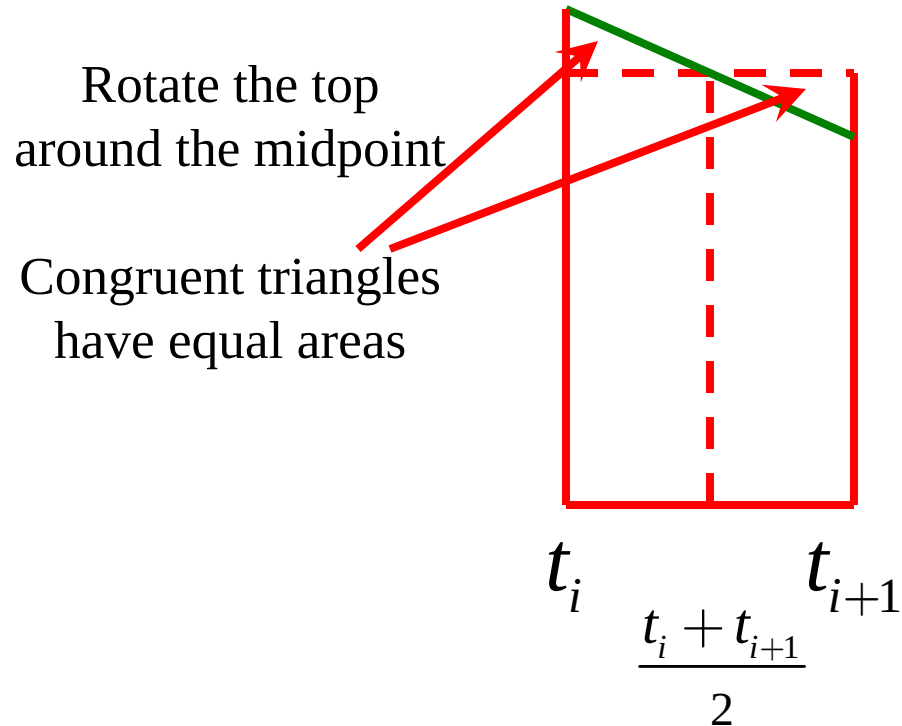
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Rotate the top
around the midpoint



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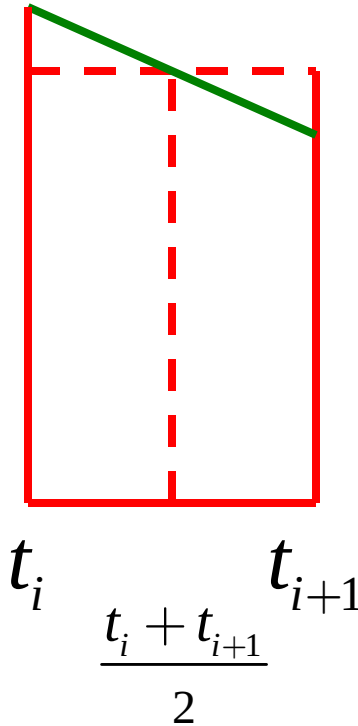


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Rotate the top
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Congruent triangles
have equal areas



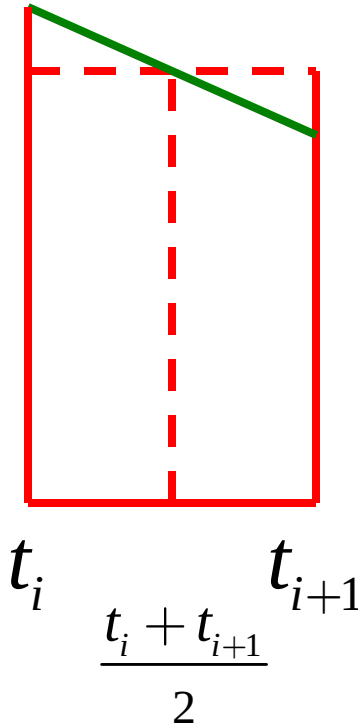
The created trapezoid has
the *same area* as the
original rectangle

Midpoint Rule

$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$

Rotate the top
around the midpoint

Congruent triangles
have equal areas

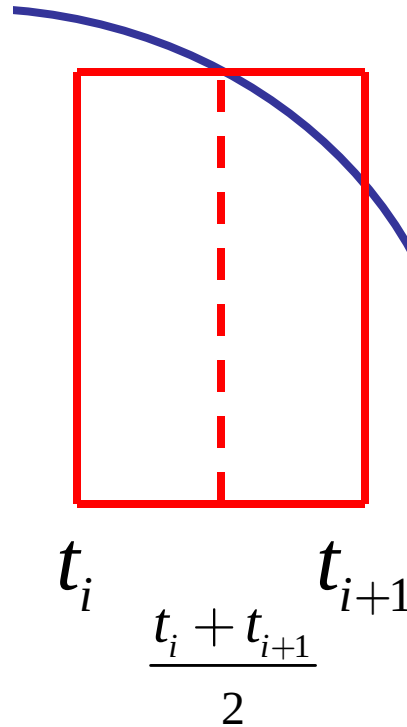


The created trapezoid has
the *same area* as the
original rectangle

... no matter what the
rotation angle is !

Midpoint Rule

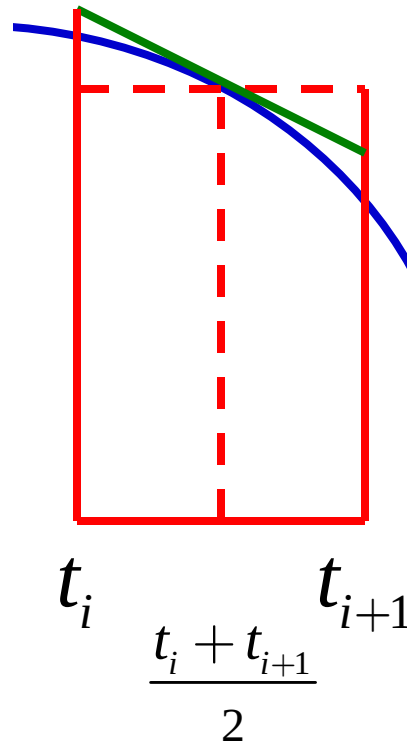
$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



Rotate the top so as to
become *tangent to the curve*

Midpoint Rule

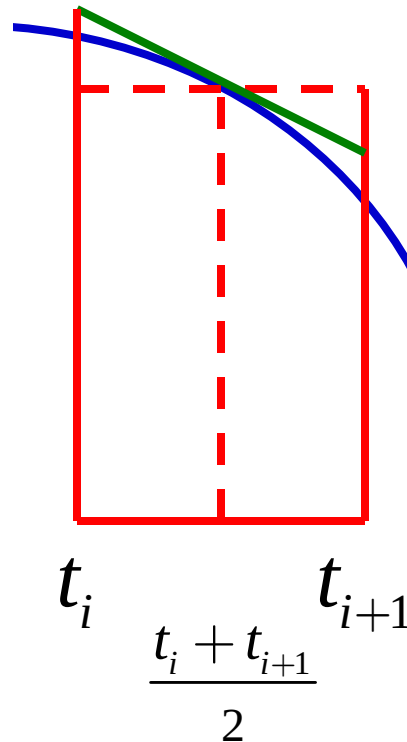
$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



Rotate the top so as to
become *tangent to the curve*

Midpoint Rule

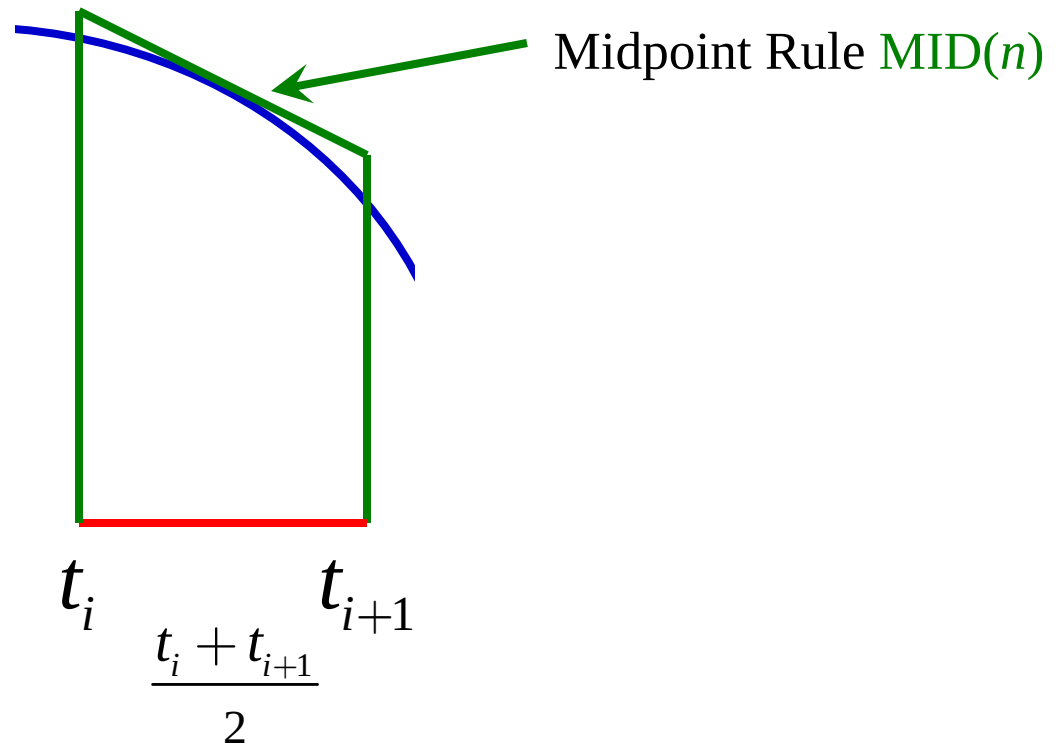
$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



The area of “tangential” trapezoid
equals
the area of the “midpoint” rectangle

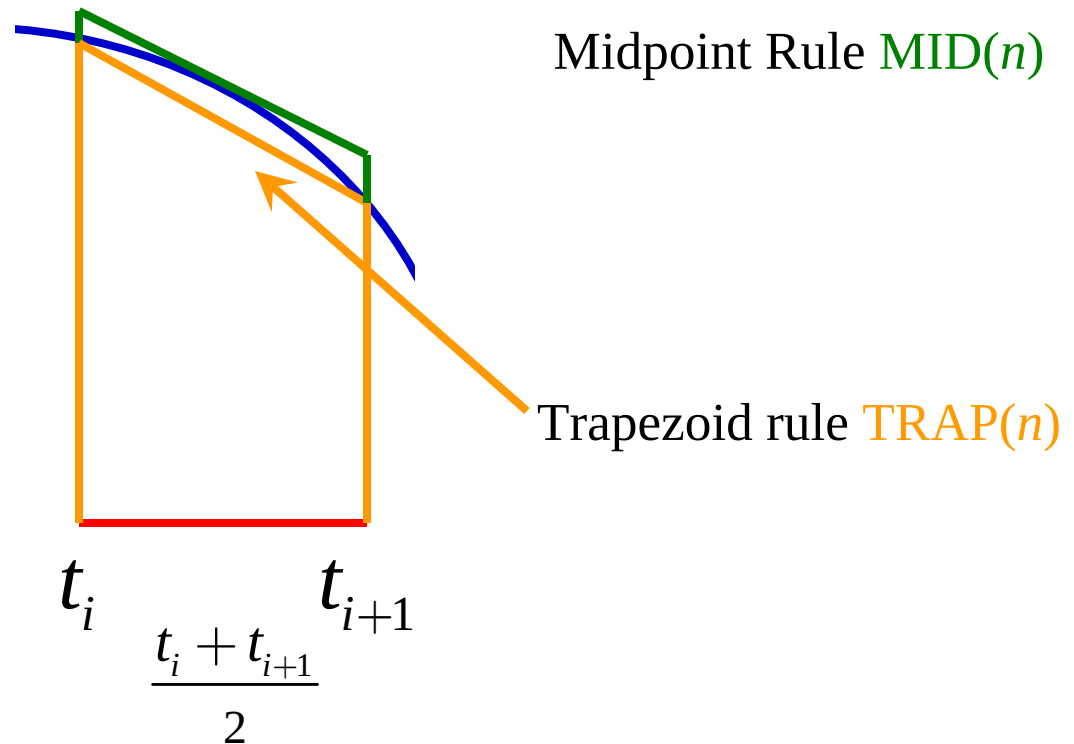
Midpoint Rule

$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



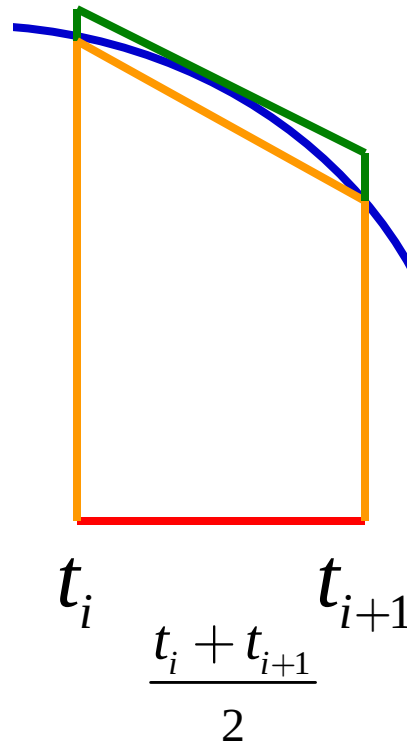
Midpoint Rule

$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



Midpoint Rule

$$\int_a^b f(t) dt \approx \sum_{i=0}^{n-1} f\left(\frac{t_i + t_{i+1}}{2}\right) \Delta t = \text{MID}(n)$$



Midpoint Rule **MID**(n)

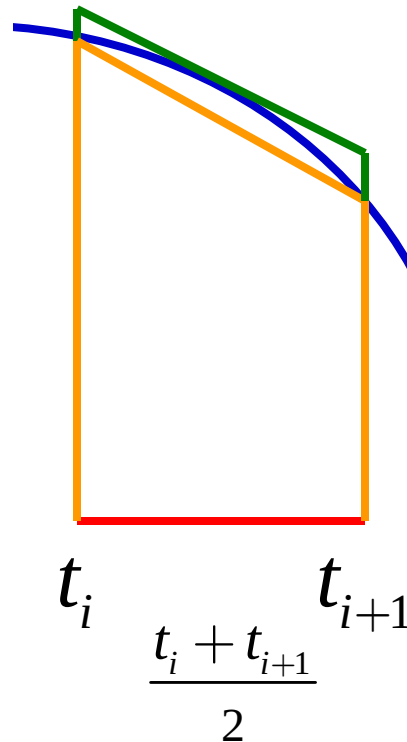
“tangent” trapezoid

Trapezoid rule **TRAP**(n)

“secant” trapezoid

Midpoint Rule

If $f(t)$ is *concave down* then: $\text{TRAP}(n) \leq \int_a^b f(t) dt \leq \text{MID}(n)$



Midpoint Rule $\text{MID}(n)$

“tangent” trapezoid

Trapezoid rule $\text{TRAP}(n)$

“secant” trapezoid

Midpoint & Trapezoid Rules

If $f(t)$ is *concave down* then: $\text{TRAP}(n) \leq \int_a^b f(t) dt \leq \text{MID}(n)$

If $f(t)$ is *concave up* then: $\text{MID}(n) \leq \int_a^b f(t) dt \leq \text{TRAP}(n)$

Error Bounds

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

We can use the Trapezoid Rule (it uses only endpoint data),

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

We can use the Trapezoid Rule (it uses only endpoint data),
but we can't use the Midpoint Rule (no data at midpoints).

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

Recall that assuming A is decreasing :

$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

$$0.86 \leq \int_5^{10} A(t) dt \leq 1.13$$

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

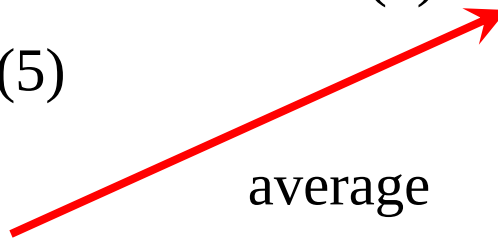
Recall that assuming A is decreasing :

$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

$$0.86 \leq \int_5^{10} A(t) dt \leq 1.13$$

$$\text{TRAP}(5) = \frac{0.86 + 1.13}{2} = 0.995$$

average



Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

Recall :

$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

$$0.86 \leq \int_5^{10} A(t) dt \leq 1.13$$

$$\text{TRAP}(5) = \frac{0.86 + 1.13}{2} = 0.995$$

If we assume $A(t)$ is concave up, then

$$\int_5^{10} A(t) dt \leq 0.995$$

Estimate the cumulative dose during the last 5 hours.

$$\int_5^{10} A(t) dt \approx ?$$

Time t (hour)						5	6	7	8	9	10
Amount A (mg/ml)						0.36	0.28	0.21	0.16	0.12	0.09

Recall :

$$\text{TRAP}(5) = \frac{0.86 + 1.13}{2} = 0.995$$

$$\text{RIGHT}(5) \leq \int_5^{10} A(t) dt \leq \text{LEFT}(5)$$

If we assume $A(t)$ is concave up, then

$$0.86 \leq \int_5^{10} A(t) dt \leq 1.13$$

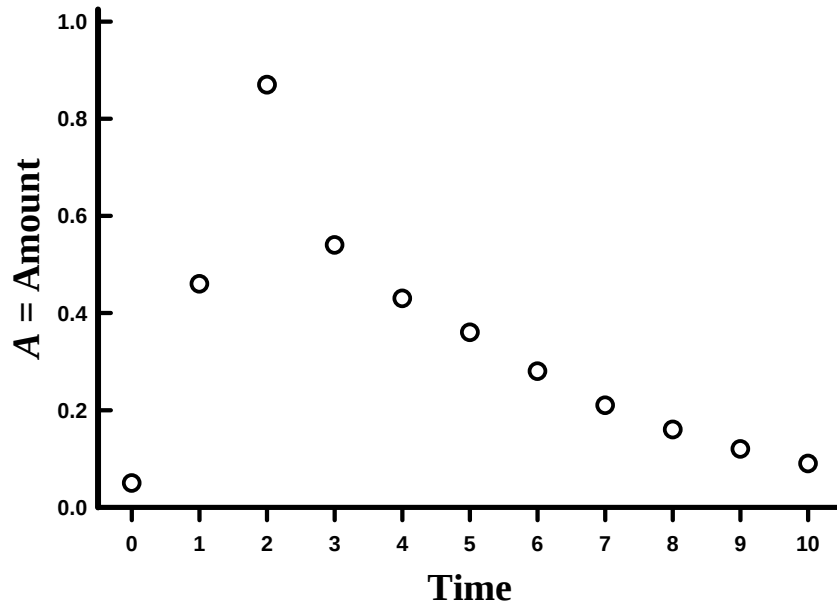
$$\int_5^{10} A(t) dt \leq 0.995$$



$$0.86 \leq \int_5^{10} A(t) dt \leq 0.995$$



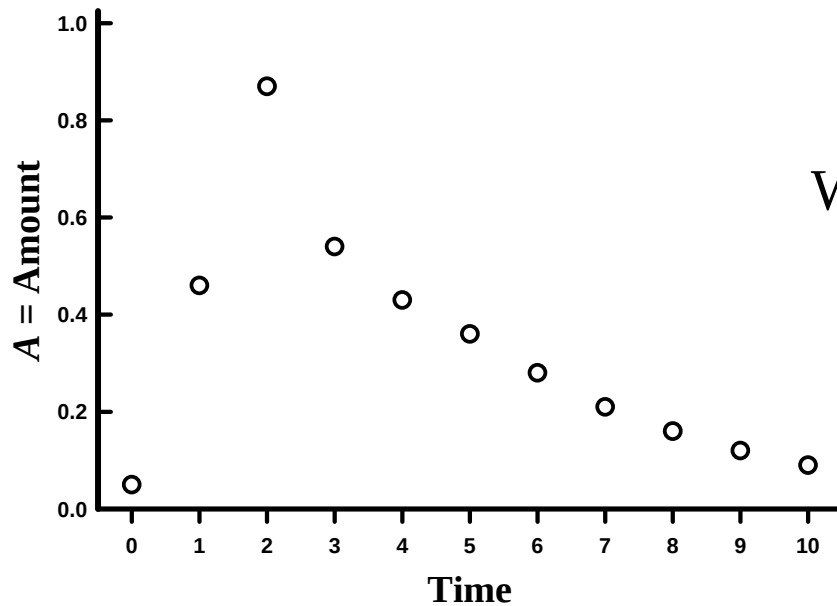
Data



$$\int_0^{10} A(t) dt \approx ?$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

Data

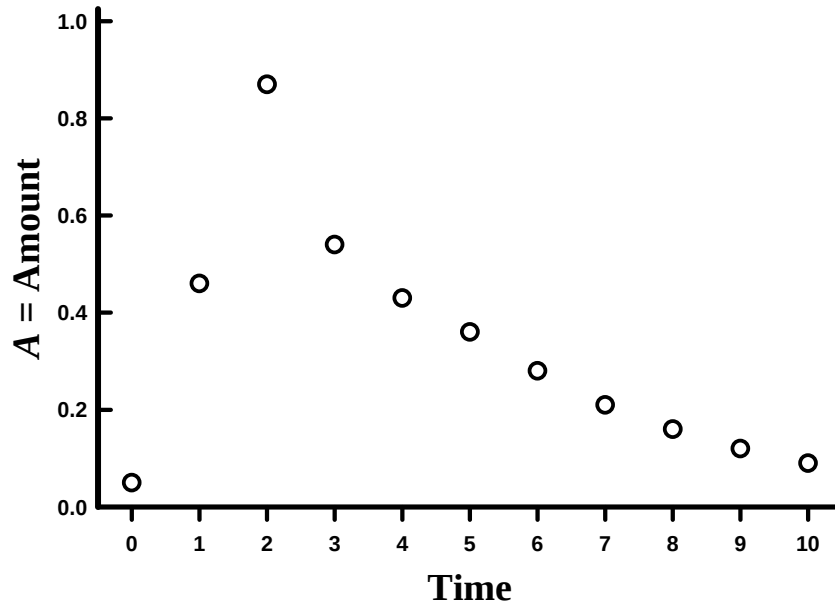


$$\int_0^{10} A(t) dt \approx ?$$

We can use rectangle & trapezoid rules,
but the error bounds don't apply

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

Data

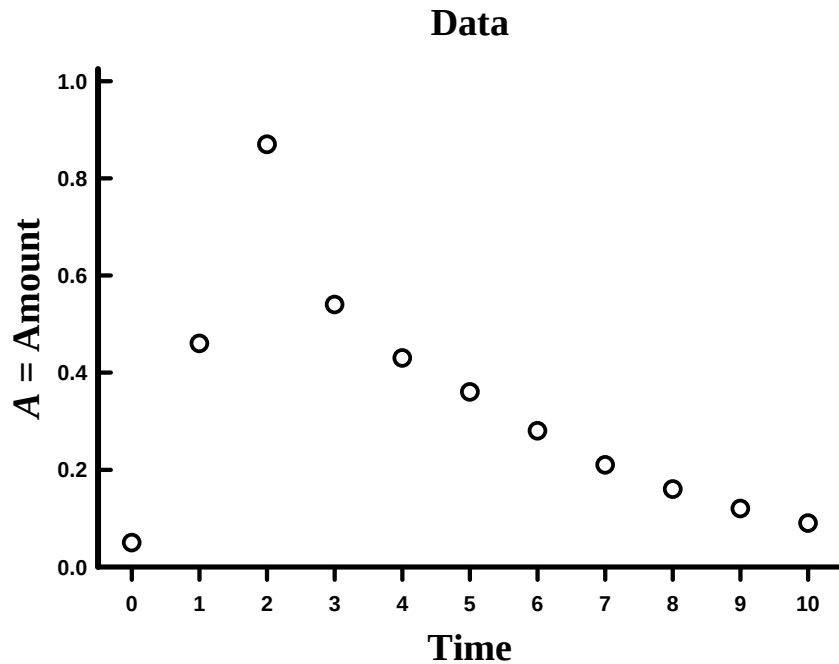


$$\int_0^{10} A(t) dt \approx ?$$

$$LEFT(10) =$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

$$t_{i+1} - t_i = 1$$

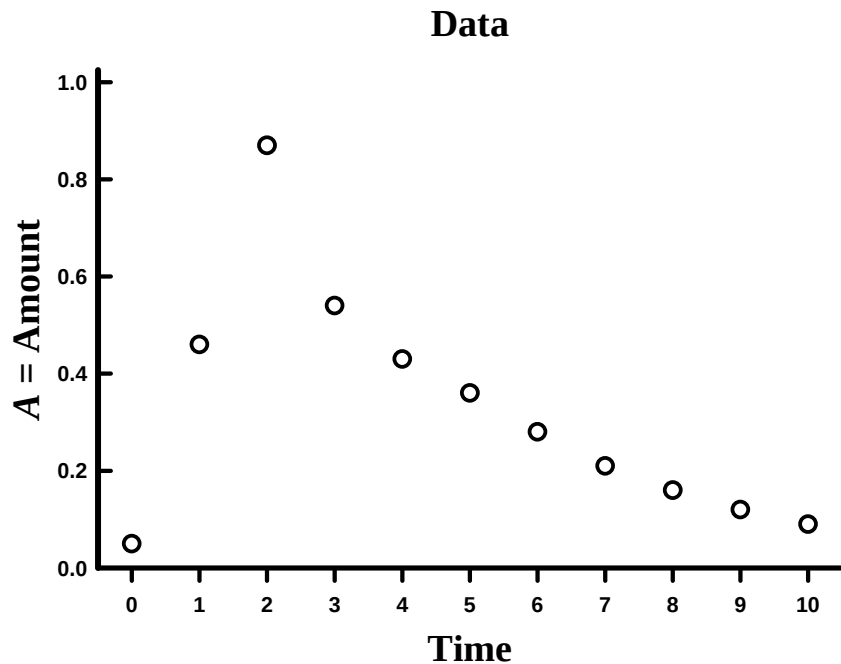


$$\int_0^{10} A(t) dt \approx ?$$

$$LEFT(10) = 3.48$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

$$t_{i+1} - t_i = 1$$



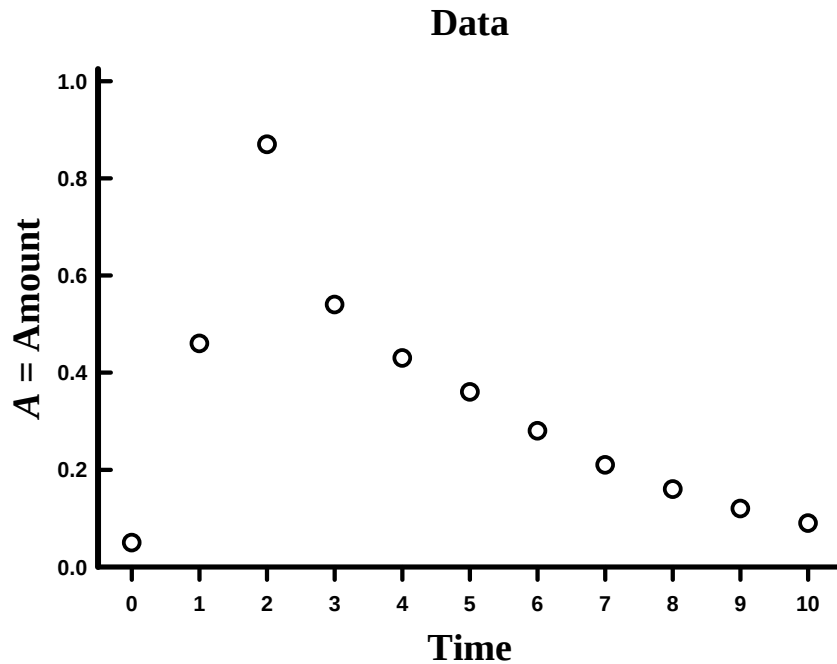
$$\int_0^{10} A(t) dt \approx ?$$

$$LEFT(10) = 3.48$$

$$RIGHT(10) = 3.52$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09

$$t_{i+1} - t_i = 1$$



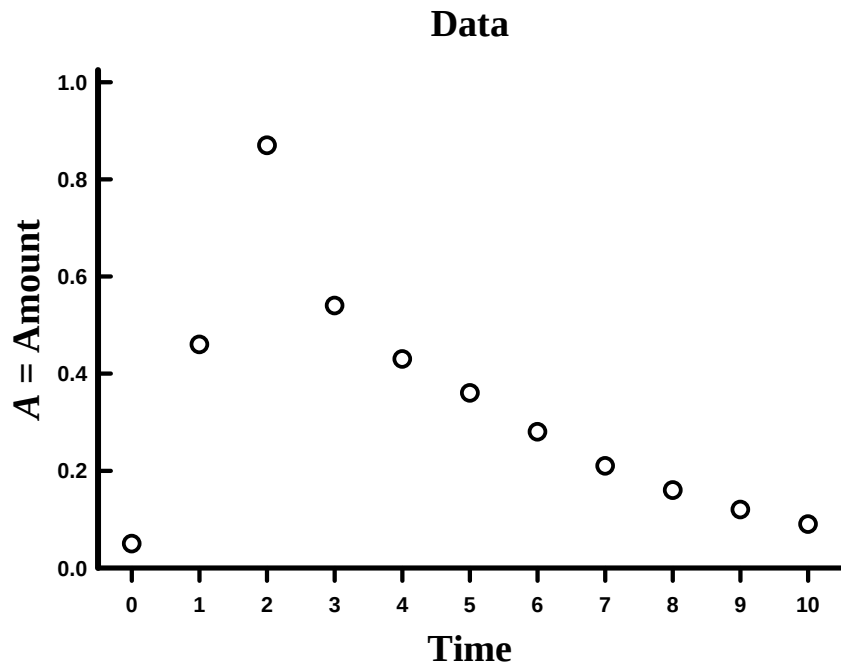
$$\int_0^{10} A(t) dt \approx ?$$

$$LEFT(10) = 3.48$$

$$RIGHT(10) = 3.52$$

$$TRAP(10) = 3.50$$

Time t (hour)	0	1	2	3	4	5	6	7	8	9	10
Amount A (mg/ml)	0.05	0.46	0.87	0.54	0.43	0.36	0.28	0.21	0.16	0.12	0.09



$$\int_0^{10} A(t) dt \approx ?$$

$$LEFT(10) = 3.48$$

$$RIGHT(10) = 3.52$$

$$TRAP(10) = 3.50$$

Because the data lacks monotonicity and/or concavity uniformly across the interval,
the theoretical error bounds do not apply.

We can't relate any of these approximations to the exact value of the integral.