

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, September 27

<http://math.arizona.edu/~cushing/250a.html>

Chapter 7: Integration

Sections 1 - 4

- Calculation of anti-derivatives (integrals)
 - ✓ 1. Learn some fundamental properties of integrals
 - ✓ 2. Use known differentiation formulas “backwards”
 - ✓ 3. Use differentiation rules “backwards”
 - chain rule $\longleftrightarrow$ substitution
 - product rule $\longleftrightarrow$ integration by parts
 - 4. Use algebraic manipulations
e.g., partial fraction decomposition
 - 5. Use tables of integrals

4. Use algebraic manipulations: Partial Fraction Decomposition
(Section 7.4 in text)

4. Use algebraic manipulations: Partial Fraction Decomposition

Useful for integrating rational functions.

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First, we look at some simpler rational functions :

$$\int \frac{1}{x-a} dx$$

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$$w = x - a,$$

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$$\int \frac{1}{x-a} dx$$

$$w = x - a, \quad \frac{dw}{dx} = 1 \Rightarrow dw = dx$$

4. Use algebraic manipulations: Partial Fraction Decomposition

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First, we look at some simpler rational functions :

$$\int \frac{1}{x-a} dx = \int \frac{1}{w} dw$$

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First, we look at some simpler rational functions :

$$\int \frac{1}{x-a} dx = \int \frac{1}{w} dw = \ln |w| + c$$

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$$\int \frac{1}{x-a} dx = \ln |x-a| + c \qquad \int \frac{1}{(x-a)^n} dx$$

for an integer $n > 1$

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$$\int \frac{1}{(x-a)^n} dx :$$

$$w = x - a, \quad \frac{dw}{dx} = 1 \Rightarrow dw = dx$$

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$$\int \frac{1}{(x-a)^n} dx = \int \frac{1}{w^n} dw$$

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$$\int \frac{1}{x-a} dx = \ln |x-a| + c \qquad \int \frac{1}{(x-a)^n} dx = \int \frac{1}{w^n} dw = \frac{1}{1-n} w^{1-n} + c$$

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$$\int \frac{1}{x^2 + a^2} dx$$

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$$\int \frac{1}{x^2 + a^2} dx$$

Recall: $\int \frac{1}{x^2 + 1} dx = \arctan x + c$

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$$\int \frac{1}{x^2 + a^2} dx = \int \frac{1}{a^2 \left(\frac{x^2}{a^2} + 1 \right)} dx$$

Recall: $\int \frac{1}{x^2 + 1} dx = \arctan x + c$

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$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a^2} \int \frac{1}{\left(\frac{x}{a}\right)^2 + 1} dx$$

$$\text{Recall: } \int \frac{1}{x^2 + 1} dx = \arctan x + c$$

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$$w = \frac{x}{a}, \quad \frac{dw}{dx} = \frac{1}{a} \Rightarrow dx = a dw$$

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$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \int \frac{1}{u^2 + 1} du$$

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$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \int \frac{1}{u^2 + 1} du = \frac{1}{a} \arctan u + c$$

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$$\int \frac{x}{x^2 + a^2} dx$$

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$$\int \frac{x}{x^2 + a^2} dx$$

$$w = x^2 + a^2, \quad \frac{dw}{dx} = 2x \Rightarrow x dx = \frac{1}{2} dw$$

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$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{x}{x^2 + a^2} dx = \frac{1}{2} \int \frac{1}{w} dw$$

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$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{x}{x^2 + a^2} dx = \frac{1}{2} \int \frac{1}{w} dw = \frac{1}{2} \ln |w| + c$$

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$$\int \frac{x}{x^2+a^2} dx = \frac{1}{2} \ln |x^2+a^2| + c$$

Note: denominator has single factor of $x-a$

(has a simple root $x=a$)

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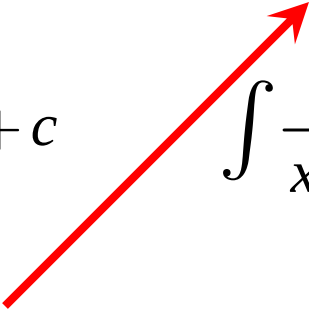
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Note: denominator has n factors of $x-a$
(has a multiple root $x=a$)

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Note: denominator has *no* (real) roots

(is not factorable)

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\int \left(\frac{2}{x-1} - \frac{3}{x^2+1} \right) dx$$

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\int \left(\frac{2}{x-1} - \frac{3}{x^2+1} \right) dx = 2 \int \frac{1}{x-1} dx - 3 \int \frac{1}{x^2+1} dx$$

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\begin{aligned}\int \left(\frac{2}{x-1} - \frac{3}{x^2+1} \right) dx &= 2 \int \frac{1}{x-1} dx - 3 \int \frac{1}{x^2+1} dx \\ &= 2 \ln |x-1| - 3 \arctan x + c\end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\int \frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} dx$$

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$$\int \frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} dx$$

Note: $\frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} = \frac{2}{x - 1} - \frac{3}{x^2 + 1}$

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\int \frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} dx = \int \left(\frac{2}{x-1} - \frac{3}{x^2+1} \right) dx$$

Note: $\frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} = \frac{2}{x-1} - \frac{3}{x^2+1}$

4. Use algebraic manipulations: Partial Fraction Decomposition

$$\int \frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} dx = \int \left(\frac{2}{x-1} - \frac{3}{x^2 + 1} \right) dx$$
$$= 2 \ln |x-1| - 3 \arctan x + c$$

Note: $\frac{2x^2 - 3x + 5}{x^3 - x^2 + x - 1} = \frac{2}{x-1} - \frac{3}{x^2 + 1}$



Called the *Partial Fraction Decomposition*

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

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Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

$$\frac{P(x)}{Q(x)} = \frac{?}{?} + \frac{?}{?} + \cdots + \frac{?}{?}$$

Want denominators of the form

$$x - a \quad \text{and} \quad x^2 + a^2.$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

1. When $\deg P(x) < \deg Q(x)$:

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

1. When $\deg P(x) < \deg Q(x)$:

- Factor the denominator $Q(x)$.

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

1. When $\deg P(x) < \deg Q(x)$:

- Factor the denominator $Q(x)$.
- A factor of the form $(x - c)$ contributes $\frac{A}{(x - c)}$.

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

1. When $\deg P(x) < \deg Q(x)$:

- Factor the denominator $Q(x)$.

- A factor of the form $(x - c)$ contributes $\frac{A}{(x - c)}$.

- A factor of the form $(x - c)^n$ contributes $\frac{A_1}{(x - c)} + \frac{A_2}{(x - c)^2} + \cdots + \frac{A_n}{(x - c)^n}$.

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

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- A factor of the form $(x^2 + a^2)$ contributes $\frac{Ax + B}{x^2 + a^2}$.

4. Use algebraic manipulations: Partial Fraction Decomposition

Strategy for calculating the *Partial Fraction Decomposition* of a rational function $\frac{P(x)}{Q(x)}$.

1. When $\deg P(x) < \deg Q(x)$:

- Factor the denominator $Q(x)$.
- A factor of the form $(x - c)$ contributes $\frac{A}{(x - c)}$.
- A factor of the form $(x - c)^n$ contributes $\frac{A_1}{(x - c)} + \frac{A_2}{(x - c)^2} + \cdots + \frac{A_n}{(x - c)^n}$.
- A factor of the form $(x^2 + a^2)$ contributes $\frac{Ax + B}{x^2 + a^2}$.

2. When $\deg P(x) \geq \deg Q(x)$, do long division first and then apply rules above to the remainder term.

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x - 1)(x + 1)}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{\boxed{(x-1)}(x+1)} = \frac{A}{x-1}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{A}{x+1}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

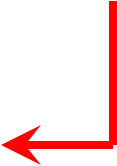
$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

How to find A_1 and A_2 ?

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{1}{x^2 - 1} = \frac{A_1(x+1) + A_2(x-1)}{(x-1)(x+1)}$$


sum up

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{1}{x^2 - 1} = \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2 - 1}$$

Choose A_1 and A_2 so that numerators are identical.

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{1}{x^2 - 1} = \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2 - 1}$$

$$\left. \begin{array}{l} A_1 + A_2 = 0 \\ A_1 - A_2 = 1 \end{array} \right\}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

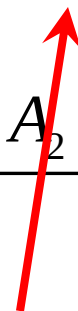
$$\frac{1}{x^2 - 1} = \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2 - 1}$$

$$\left. \begin{array}{l} A_1 + A_2 = 0 \\ A_1 - A_2 = 1 \end{array} \right\} \Rightarrow \begin{array}{l} A_1 = 1/2 \\ A_2 = -1/2 \end{array}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{A_1}{x-1} + \frac{A_2}{x+1}$$

$$\frac{1}{x^2 - 1} = \frac{(A_1 + A_2)x + (A_1 - A_2)}{x^2 - 1}$$


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4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1}$$

$$\int \frac{1}{x^2 - 1} dx = \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1}$$

$$\begin{aligned} \int \frac{1}{x^2 - 1} dx &= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx \\ &= \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + c \end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{1}{x^2 - 1} = \frac{1}{(x-1)(x+1)} = \frac{1}{2} \frac{1}{x-1} - \frac{1}{2} \frac{1}{x+1}$$

$$\begin{aligned} \int \frac{1}{x^2 - 1} dx &= \frac{1}{2} \int \frac{1}{x-1} dx - \frac{1}{2} \int \frac{1}{x+1} dx \\ &= \frac{1}{2} \ln |x-1| - \frac{1}{2} \ln |x+1| + c \\ &= \frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + c \end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x + 2)(x^2 + 3)}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

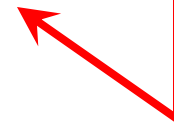
$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x+2)(x^2+3)} = \frac{A_1}{x+2} + \frac{A_2x + B_2}{x^2+3}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x+2)(x^2+3)} = \boxed{\frac{A_1}{x+2} + \frac{A_2x + B_2}{x^2+3}}$$

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{(A_1 + A_2)x^2 + (2A_2 + B_2)x + (3A_1 + 2B_2)}{(x+2)(x^2+3)}$$



sum up

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x+2)(x^2+3)} = \frac{A_1}{x+2} + \frac{A_2x + B_2}{x^2+3}$$

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{(A_1 + A_2)x^2 + (2A_2 + B_2)x + (3A_1 + 2B_2)}{x^3 + 2x^2 + 3x + 6}$$

Equate numerators

$$A_1 + A_2 = -1$$

$$2A_2 + B_2 = 1$$

$$3A_1 + 2B_2 = -8$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x+2)(x^2+3)} = \frac{A_1}{x+2} + \frac{A_2x + B_2}{x^2+3}$$

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{(A_1 + A_2)x^2 + (2A_2 + B_2)x + (3A_1 + 2B_2)}{(x+2)(x^2+3)}$$

Equate numerators

$$A_1 + A_2 = -1$$

$$A_1 = -2$$

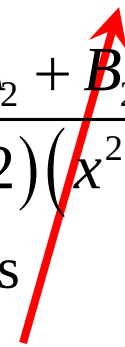
$$2A_2 + B_2 = 1 \quad \Rightarrow \quad A_2 = 1$$

$$3A_1 + 2B_2 = -8 \quad B_2 = -1$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{-x^2 + x - 8}{(x+2)(x^2+3)} = \frac{-2}{x+2} + \frac{x-1}{x^2+3}$$

$$\frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} = \frac{(A_1 + A_2)x^2 + (2A_2 + B_2)x + (3A_1 + 2B_2)}{(x+2)(x^2+3)}$$


Equate numerators

$$A_1 + A_2 = -1$$

$$A_1 = -2$$

$$2A_2 + B_2 = 1 \quad \Rightarrow \quad A_2 = 1$$

$$3A_1 + 2B_2 = -8 \quad B_1 = -1$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\int \frac{-x^2 + x - 8}{x^3 + 2x^2 - 3x + 6} dx = \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\begin{aligned}\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx &= \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx \\ &= -2 \int \frac{1}{x+2} dx + \int \frac{x-1}{x^2+3} dx\end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\begin{aligned}\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx &= \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx \\ &= -2 \int \frac{1}{x+2} dx + \int \left(\frac{x}{x^2+3} - \frac{1}{x^2+3} \right) dx\end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\begin{aligned}\int \frac{-x^2 + x - 8}{x^3 + 2x^2 - 3x + 6} dx &= \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx \\ &= -2 \int \frac{1}{x+2} dx + \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx\end{aligned}$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx = \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx$$

$$= -2 \int \frac{1}{x+2} dx + \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx$$

$$= -2 \ln|x+2|$$

$$w = x + 2$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx = \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx$$

$$= -2 \int \frac{1}{x+2} dx + \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx$$

$$= -2 \ln|x+2| + \frac{1}{2} \ln|x^2+3|$$

$$w = x^2 + 3$$

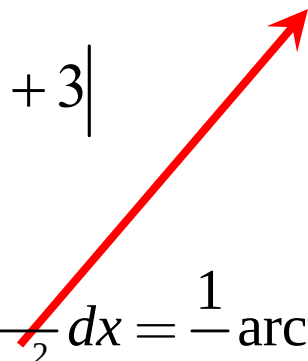
4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx = \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx$$

$$= -2 \int \frac{1}{x+2} dx + \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx$$

$$= -2 \ln|x+2| + \frac{1}{2} \ln|x^2+3|$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$


$$a^2 = 3 = (\sqrt{3})^2$$

4. Use algebraic manipulations: Partial Fraction Decomposition

Example

$$\begin{aligned}\int \frac{-x^2 + x - 8}{x^3 + 2x^2 + 3x + 6} dx &= \int \left(\frac{-2}{x+2} + \frac{x-1}{x^2+3} \right) dx \\ &= -2 \int \frac{1}{x+2} dx + \int \frac{x}{x^2+3} dx - \int \frac{1}{x^2+3} dx \\ &= -2 \ln|x+2| + \frac{1}{2} \ln|x^2+3| - \frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + c\end{aligned}$$

Chapter 7: Integration

Sections 1 - 4

- Calculation of anti-derivatives (integrals)
 - ✓ 1. Learn some fundamental properties of integrals
 - ✓ 2. Use known differentiation formulas “backwards”
 - ✓ 3. Use differentiation rules “backwards”
 - chain rule $\longleftrightarrow$ substitution
 - product rule $\longleftrightarrow$ integration by parts
 - ✓ 4. Use algebraic manipulations
 - e.g., partial fraction decomposition
 - 5. Use tables of integrals

5. Use of Table of Integrals

Refer to inside back cover of text.

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

These come from integration by parts ...

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

$$\int \sin 2x \sin 3x dx$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

$$\int \sin 2x \sin 3x dx = \frac{1}{3^2 - 2^2} [2 \cos 2x \sin 3x - 2 \sin 2x \cos 3x] + c$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

These come from integration by parts ...

Note the recursive nature of the formulas.

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

$$\int (x^2 - 2x + 3)e^{-x} dx$$

$$a = -1$$

$$p(x) = x^2 - 2x + 3$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

$$\int (x^2 - 2x + 3)e^{-x} dx = \frac{1}{-1} (x^2 - 2x + 3)e^{-x}$$

$$a = -1$$

$$p(x) = x^2 - 2x + 3$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

$$\int (x^2 - 2x + 3)e^{-x} dx = \frac{1}{-1} (x^2 - 2x + 3)e^{-x}$$

$$a = -1$$

$$p(x) = x^2 - 2x + 3$$

$$- \frac{1}{(-1)^2} (2x - 2)e^{-x}$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

$$\int (x^2 - 2x + 3)e^{-x} dx = \frac{1}{-1} (x^2 - 2x + 3)e^{-x}$$

$$a = -1$$

$$p(x) = x^2 - 2x + 3$$

$$- \frac{1}{(-1)^2} (2x - 2)e^{-x}$$

$$+ \frac{1}{(-1)^3} (2)e^{-x} + c$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

$$\int (x^2 - 2x + 3)e^{-x} dx = -(x^2 + 3)e^{-x} + c$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

These come from integration by parts ...

Note the recursive nature of the formulas.

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^{4-1} x \cos x + \frac{4-1}{4} \int \sin^{4-2} x \, dx$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^0 x \, dx$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \int \sin^2 x \, dx$$

$$\begin{aligned} \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^0 x \, dx \\ &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x \end{aligned}$$

5. Use of Table of Integrals

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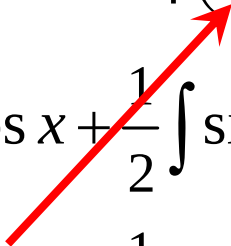
I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \left(-\frac{1}{2} \sin x \cos x + \frac{1}{2} x \right)$$

$$\begin{aligned} \int \sin^2 x \, dx &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} \int \sin^0 x \, dx \\ &= -\frac{1}{2} \sin x \cos x + \frac{1}{2} x \end{aligned}$$


5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

$$\int \sin^4 x \, dx = -\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + c$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

V. Quadratic in the Denominator

These are basic to integration by partial fraction decompositions.

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

V. Quadratic in the Denominator

VI. Integrals Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

These come from trigonometric substitutions.

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

V. Quadratic in the Denominator

VI. Integrals Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

$$\int \sqrt{3 + x^2} \, dx = \frac{1}{2} \left(x\sqrt{3 + x^2} + 3 \int \frac{1}{\sqrt{3 + x^2}} \, dx \right)$$

5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

V. Quadratic in the Denominator

VI. Integrals Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

$$\int \sqrt{3 + x^2} \, dx = \frac{1}{2} \left(x\sqrt{3 + x^2} + 3 \int \frac{1}{\sqrt{3 + x^2}} \, dx \right)$$

$$\int \frac{1}{\sqrt{3 + x^2}} \, dx = \ln \left| x + \sqrt{3 + x^2} \right|$$


5. Use of Table of Integrals

Refer to inside back cover of text.

I. Basic Functions

II. Products of e^x , $\cos x$, and $\sin x$

III. Products of Polynomial $p(x)$ with $\ln x$, e^x , $\cos x$, $\sin x$

IV. Integer Powers of $\sin x$ and $\cos x$

V. Quadratic in the Denominator

VI. Integrals Involving $\sqrt{a^2 + x^2}$, $\sqrt{a^2 - x^2}$, $\sqrt{x^2 - a^2}$, $a > 0$

$$\int \sqrt{3 + x^2} \, dx = \frac{1}{2} \left(x\sqrt{3 + x^2} + 3 \ln \left| x + \sqrt{3 + x^2} \right| \right) + c$$