

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Tuesday, September 4

<http://math.arizona.edu/~cushing/250a.html>

Chapter 1.6

Power Functions, Polynomials Functions, Rational Functions

At this point we've not only reviewed
linear and exponential/logarithmic functions,
but also other topics from Chapters 2 – 4

- Derivatives: Chapter 2, Sections 2.2 – 2.6
- Differentiation: Chapter 3, Sections 3.1 – 3.6
- Using the derivative: Chapter 4, Sections 4.1 & 4.3

Power Functions

A Power Law (or Allometric Law)

asserts a relationship between two variables y, x of the form

$$y = a x^r$$

a, r are constants

In applications,
a common problem is to calculate or estimate
the constants a, r from data.

$$y = a x^r$$

a, r are constants

Recall: $\frac{dy}{dx} = a r x^{r-1}, \quad \frac{\partial^2 y}{\partial x^2} = a r (r-1) x^{r-2}$

$$y = a x^r$$

a, r are constants

Recall: $\frac{dy}{dx} = a r x^{r-1}, \quad \frac{\partial^2 y}{\partial x^2} = a r(r-1) x^{r-2}$

EXAMPLES

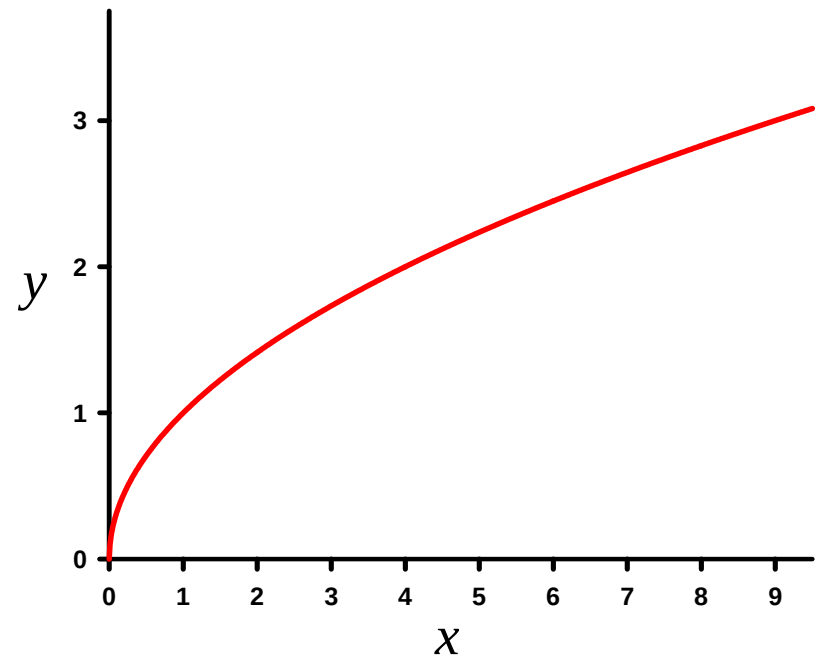
$$y = x^{1/2}, \quad x > 0$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-1/2} > 0$$

$\Rightarrow y$ is increasing

$$\frac{d^2 y}{dx^2} = -\frac{1}{4} x^{-3/2} < 0$$

$\Rightarrow y$ is concave down



$$y = a x^r$$

a, r are constants

Recall: $\frac{dy}{dx} = a r x^{r-1}, \quad \frac{\partial^2 y}{\partial x^2} = a r(r-1) x^{r-2}$

EXAMPLES

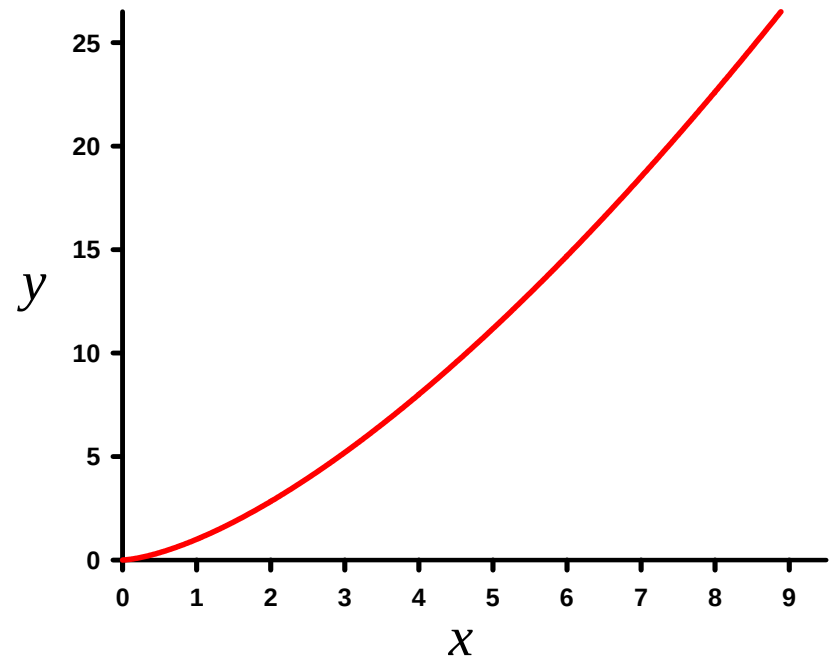
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EXAMPLES

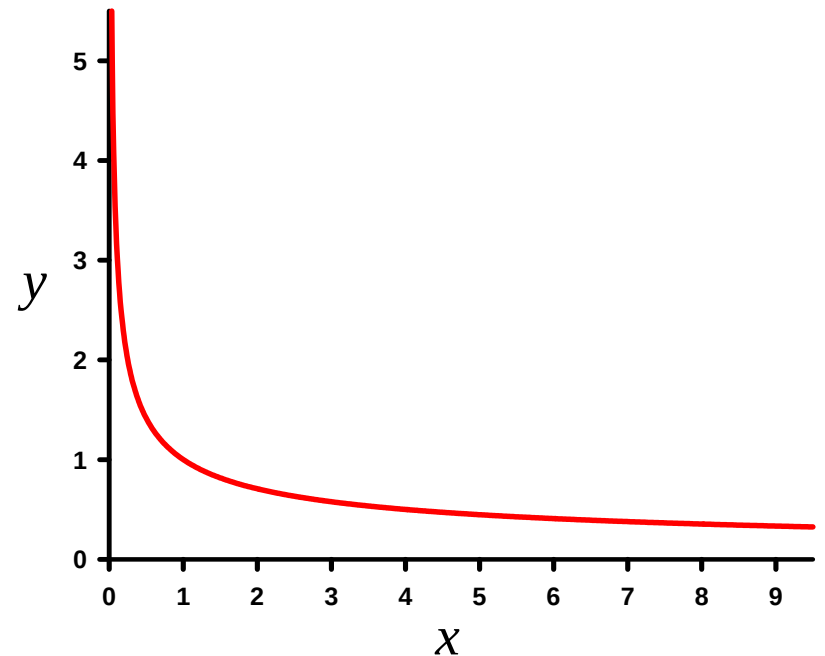
$$y = x^{-1/2}, \quad x > 0$$

$$\frac{dy}{dx} = -\frac{1}{2} x^{-3/2} < 0$$

$\Rightarrow y$ is decreasing

$$\frac{d^2 y}{dx^2} = \frac{3}{4} x^{-5/2} > 0$$

$\Rightarrow y$ is concave up



APPLICATIONS

In ecology, a power (allometric) law is often assumed to hold between the variables

y = number of species, x = habitat size (area)

$$y = a x^r$$

We can use 2 data points to calculate a and r .

And then use the law to predict y from x or vice versa.

APPLICATIONS

In ecology, a power (allometric) law is often assumed to hold between the variables

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$$y = a x^r$$

EXAMPLE

Habitat	x <i>km²</i>	y <i>species of birds</i>
Island A	15	5
Island B	110	9

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Island C	74	?
Island D	?	20

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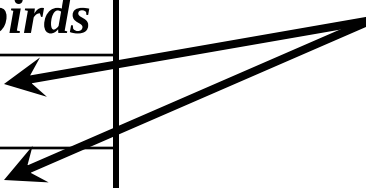
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Habitat	x <i>km²</i>	y <i>species of birds</i>
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Interpolate
these 2 data points
to calculate a and r .



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$$5 = a 15^r$$

$$9 = a 110^r$$

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$$5 = a 15^r \Rightarrow a = 5 \times 15^{-r}$$
$$9 = a 110^r$$


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Island A	15	5
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$$5 = a 15^r \Rightarrow a = 5 \times 15^{-r}$$

$$9 = a 110^r \Rightarrow 9 = 5 \times 15^{-r} \times 110^r$$

Solve for

$$r = \frac{\ln(9 / 5)}{\ln(110 / 15)}$$

APPLICATIONS

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Solve for

$$r = \frac{\ln(9/5)}{\ln(110/15)}$$

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Habitat	x <i>km²</i>	y <i>species of birds</i>
Island A	15	5
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Island C	74	?
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$$a = 5 \times 15^{-\frac{\ln(9/5)}{\ln(110/15)}} \approx 2.25$$

$$r = \frac{\ln(9/5)}{\ln(110/15)} \approx 0.295$$

APPLICATIONS

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$$y = 2.25 x^{0.295}$$

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$$y = 2.25 x^{0.295}$$

$$y = 2.25 \times 74^{0.295} \approx 8.01$$


APPLICATIONS

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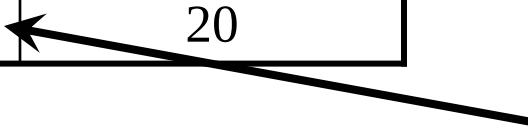
$$y = a x^r$$

EXAMPLE

Habitat	x <i>km²</i>	y <i>species of birds</i>
Island A	15	5
Island B	110	9
Island C	74	8
Island D	1648	20

$$y = 2.25 x^{0.295}$$

$$20 = 2.25 x^{0.295}$$

$$x \approx 1648$$


APPLICATIONS

$$y = a x^r$$

Another way to do allometric modeling is to convert variables to the log scale:

$$Y = \ln y, \quad X = \ln x$$

Why? Because Y and X are *linearly* related !

$$\ln y = \ln(ax^r) = r \ln x + \ln a$$

APPLICATIONS

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slope


intercept

APPLICATIONS

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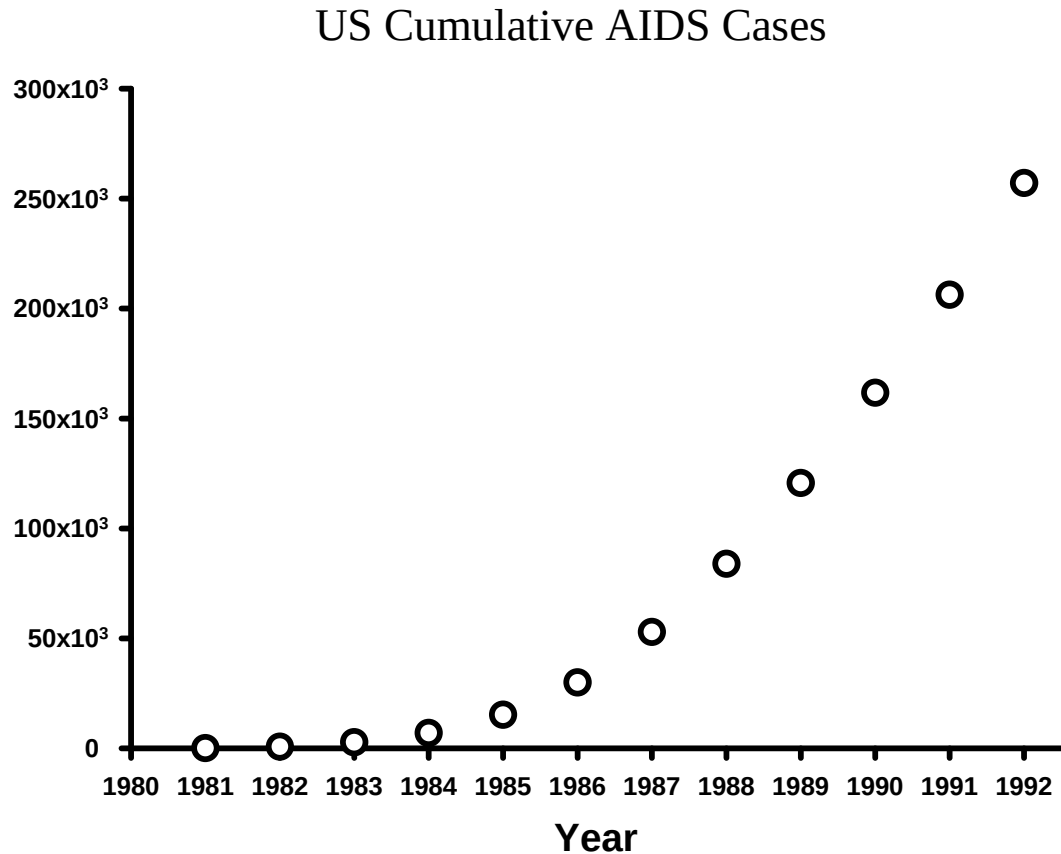
$$Y = rX + \ln a$$

We can find “best” power law fit to data using the Linear Least Squares Method *on the log scale*.

EXAMPLE

Accumulated Cases of AIDS in U.S.

<i>year</i>	<i>y</i>
1981	97
1982	709
1983	2,698
1984	6,928
1985	15,242
1986	29,944
1987	52,902
1988	83,903
1989	120,612
1990	161,711
1991	206,247
1992	257,085



EXAMPLE

Accumulated Cases of AIDS in U.S.

X	Y
0.0000	4.5747
0.6931	6.5639
1.0986	7.9003
1.3863	8.8433
1.6094	9.6318
1.7918	10.3071
1.9459	10.8762
2.0794	11.3374
2.1972	11.7003
2.3026	11.9936
2.3979	12.2368
2.4849	12.4572

x = year since 1980

y = cumulative # AIDS cases

Calculate the “best” power law fit

$$y = a x^r$$

by the Least Squares Method on the log scale

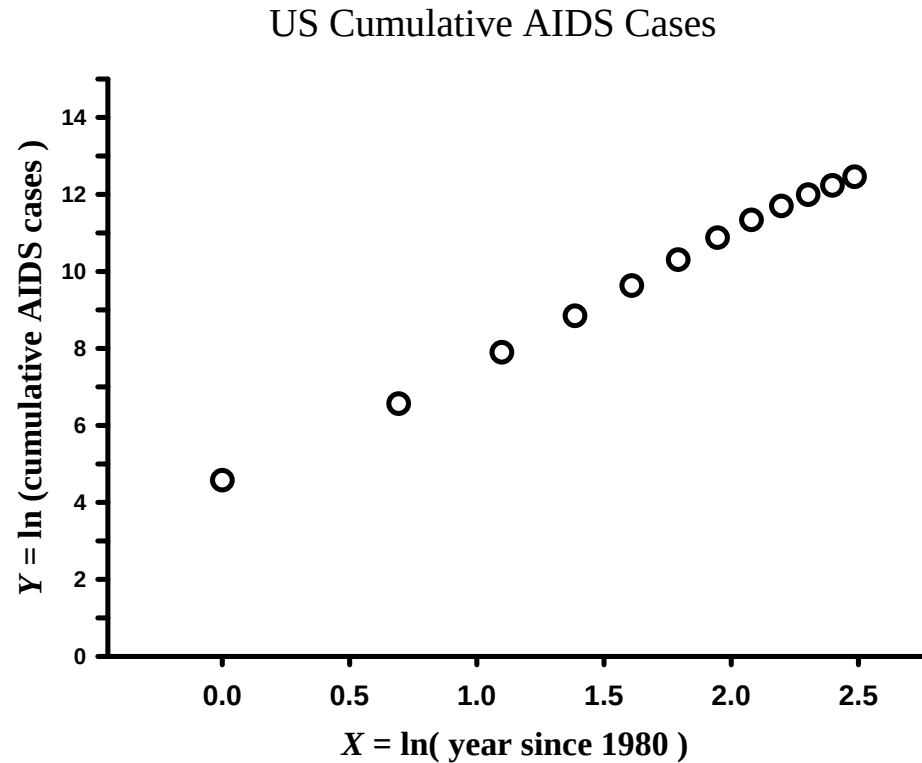
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$$X = \ln x, \quad Y = \ln y$$

EXAMPLE

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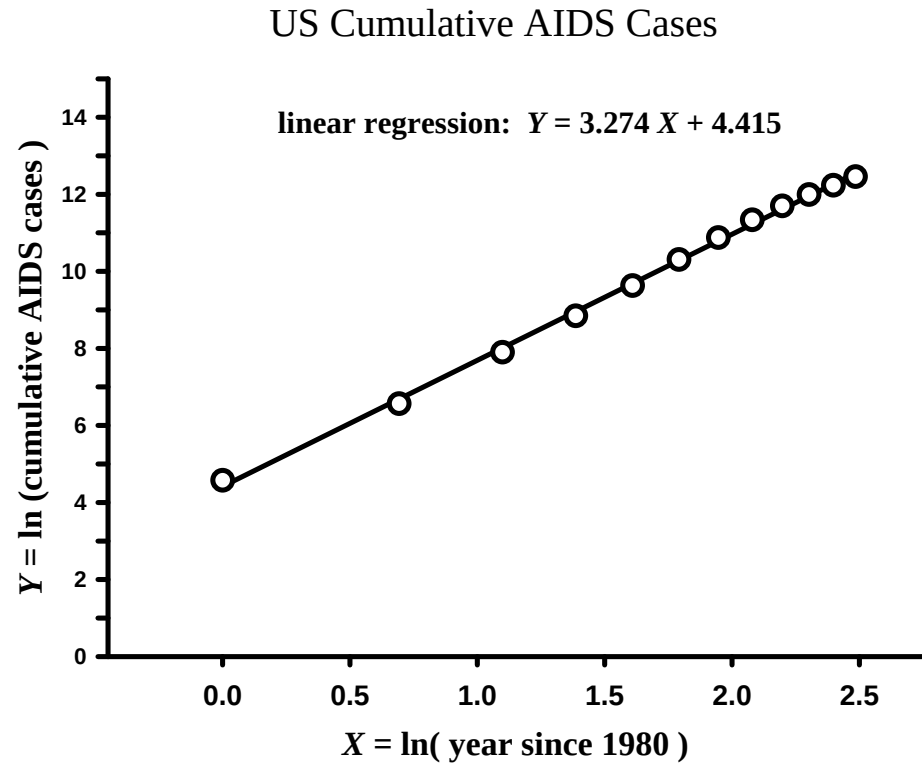
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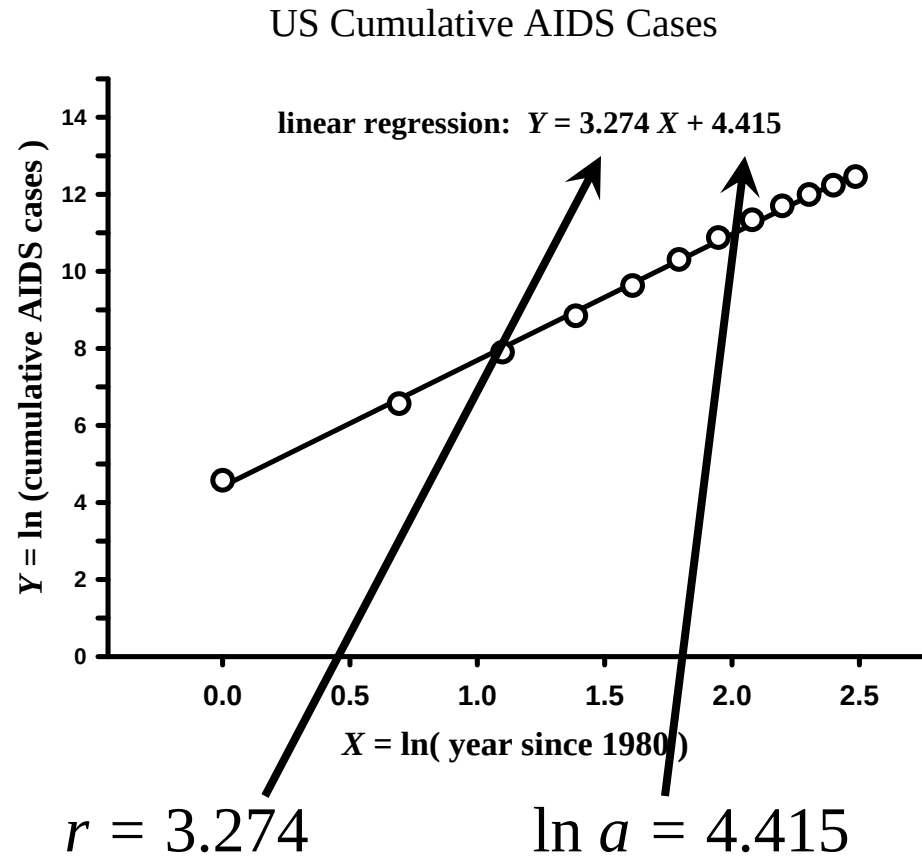
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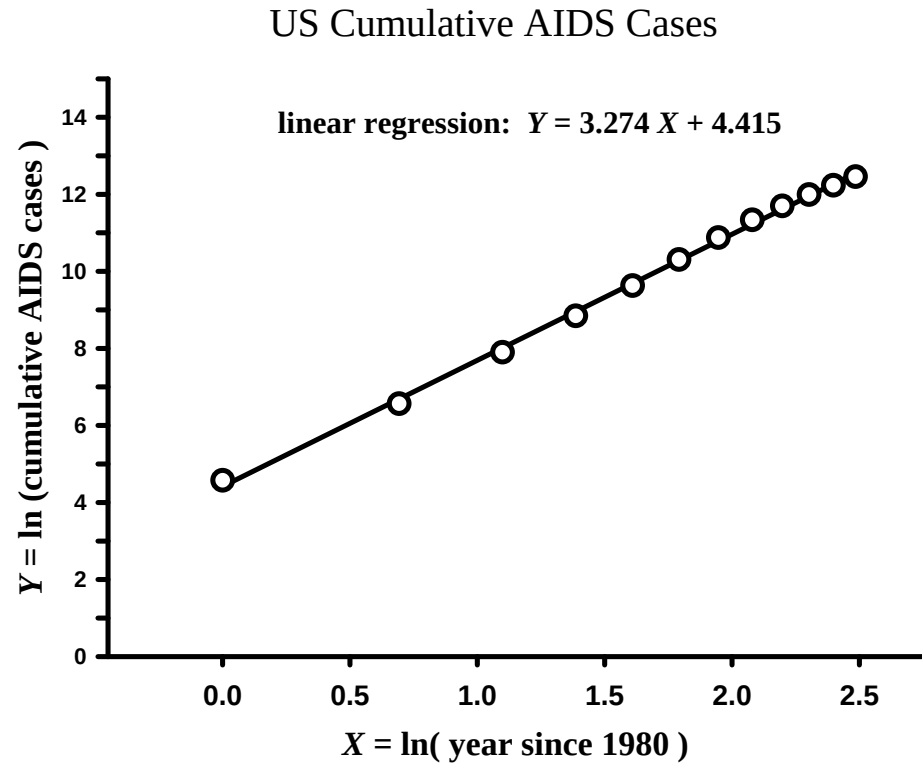
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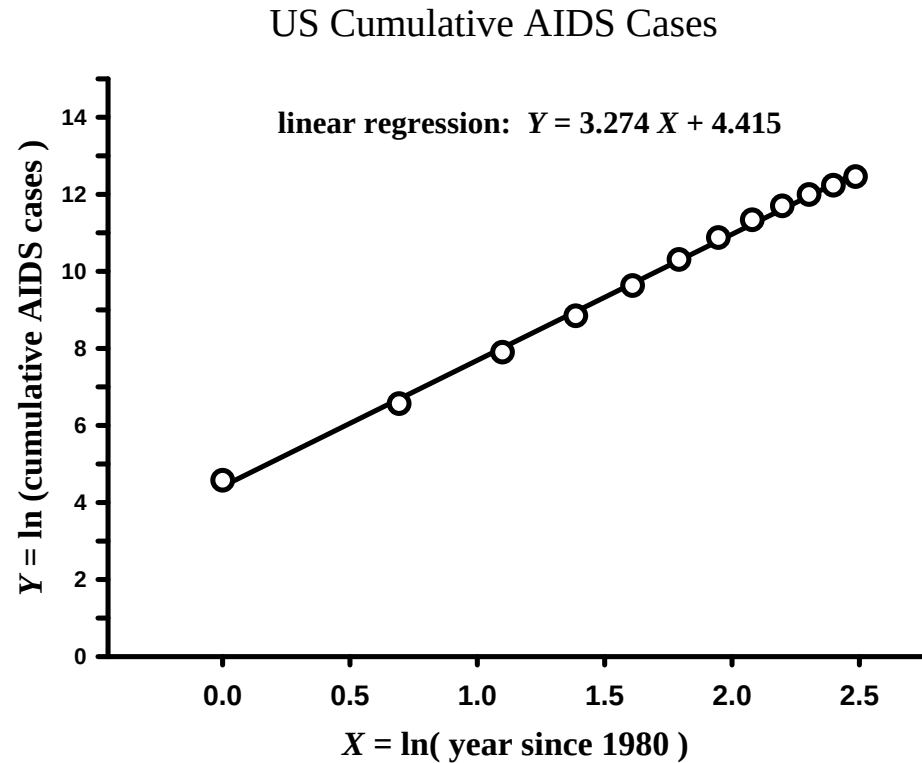
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$$y = a x^r \Rightarrow y = e^{4.415} x^{3.274}$$

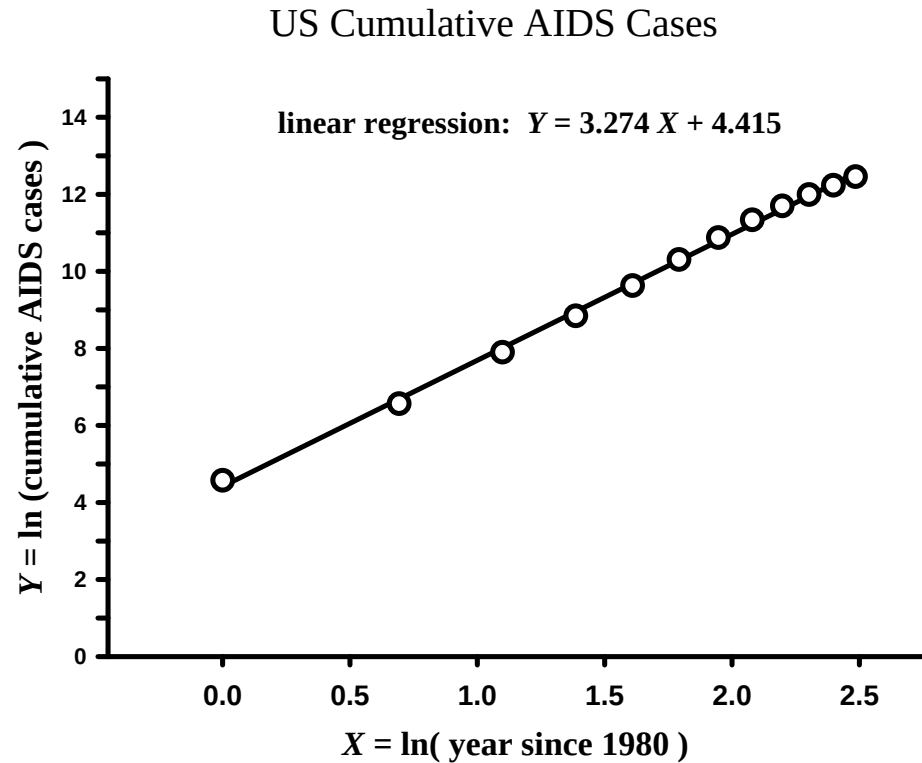
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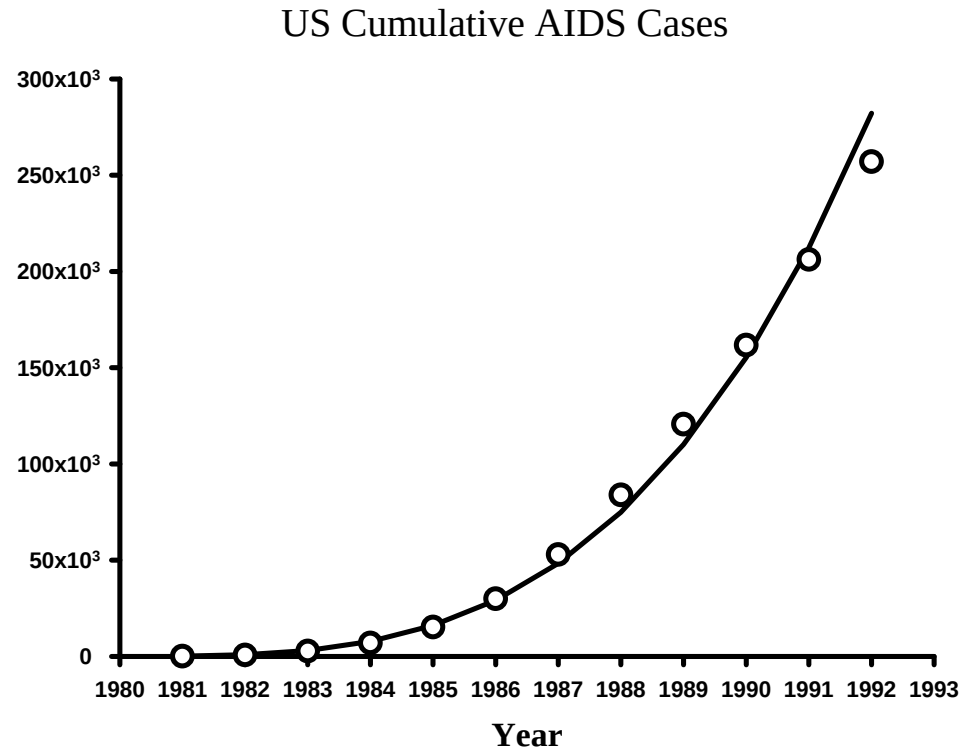


$$y = a x^r \Rightarrow y = e^{4.415} x^{3.274} \approx 82.68 x^{3.274}$$

EXAMPLE

Accumulated Cases of AIDS in U.S.

<i>year</i>	<i>y</i>
1981	97
1982	709
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$$y \approx 82.68(\text{year} - 1980)^{3.274}$$

Polynomial Functions

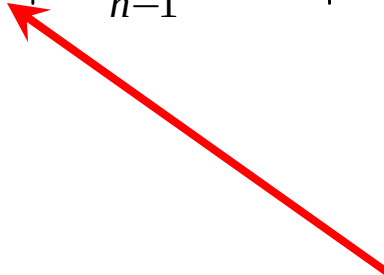
Polynomials are linear combinations of power functions with integer exponents:

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Polynomial Functions

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the degree

Polynomial Functions

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the coefficients

Polynomial Functions

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$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

Take derivatives using power rule

and the differentiation rules :

$$\frac{d(P+Q)}{dx} = \frac{dP}{dx} + \frac{dQ}{dx}, \quad \frac{d(aP)}{dx} = a \frac{dP}{dx}$$

or more concisely

$$\frac{d(a_1 P + a_2 Q)}{dx} = a_1 \frac{dP}{dx} + a_2 \frac{dQ}{dx}$$

Polynomial Functions

Polynomials are linear combinations of power functions with integer exponents:

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

$$y = a_1 x + a_0$$

first degree polynomial (linear polynomial)

geometry = straight lines

Polynomial Functions

Polynomials are linear combinations of power functions with integer exponents:

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

$$y = a_2 x^2 + a_1 x + a_0, \quad a_2 \neq 0$$

second degree polynomial (linear polynomial)

geometry = parabolas

$$\frac{dy}{dx} = 2a_2 x + a_1 = 0 \Rightarrow x = -a_1 / 2a_2 \quad \text{one critical point}$$

$$\frac{d^2 y}{dx^2} = 2a_2 \Rightarrow \begin{cases} > 0, & \text{concave up (minimum at critical point) if } a_2 > 0 \\ < 0, & \text{concave down (maximum at critical point) if } a_2 < 0 \end{cases}$$

Polynomial Functions

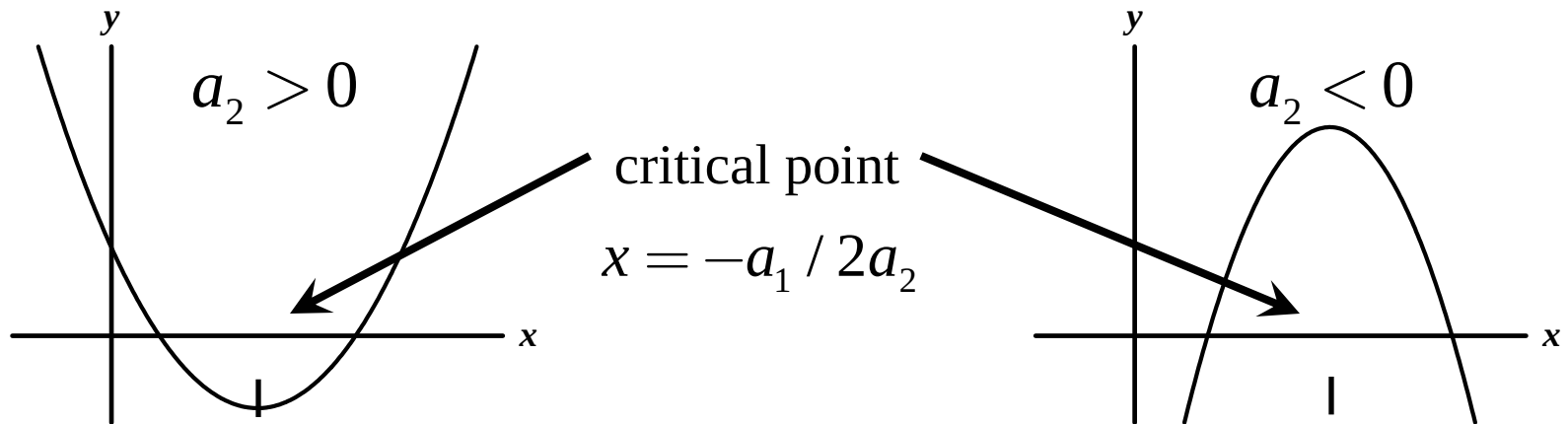
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second degree polynomial (quadratic polynomial)

geometry = parabolas



Polynomial Functions

Three data points determine a parabola

$$y(x) = a_2x^2 + a_1x + a_0$$

$$(x, y) = (-1, 1) \Rightarrow y(-1) = 1 \Rightarrow a_2 - a_1 + a_0 = 1$$

$$(x, y) = (0, 2) \Rightarrow y(0) = 2 \Rightarrow a_0 = 2$$

$$(x, y) = (1, -2) \Rightarrow y(1) = -2 \Rightarrow a_2 + a_1 + a_0 = -2$$

3 equations in 3 unknowns

Polynomial Functions

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$$a_0 = 2, \quad a_1 = -\frac{3}{2}, \quad a_2 = -\frac{5}{2}$$

$$y(x) = -\frac{5}{2}x^2 - \frac{3}{2}x + 2$$

Polynomial Functions

For more than three data points, we could determine the “best” parabolic fit by the *Nonlinear* Least Squares Method.

Polynomial Functions

An Application

In ecology, an often used law for population growth is the

Logistic Growth Law :

the *rate of change* of population density P is a function of P
as given by the formula

$$\frac{dP}{dt} = r \left(1 - \frac{P}{K} \right) P = -\frac{r}{K} P^2 + rP$$

a (downward) parabola

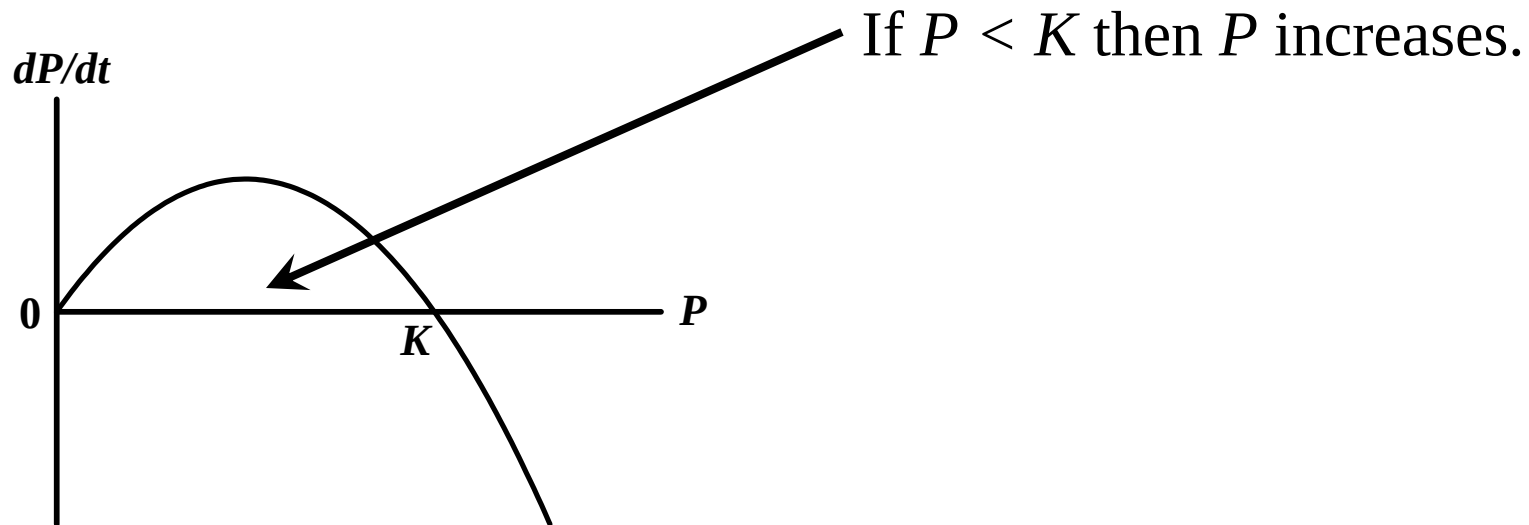
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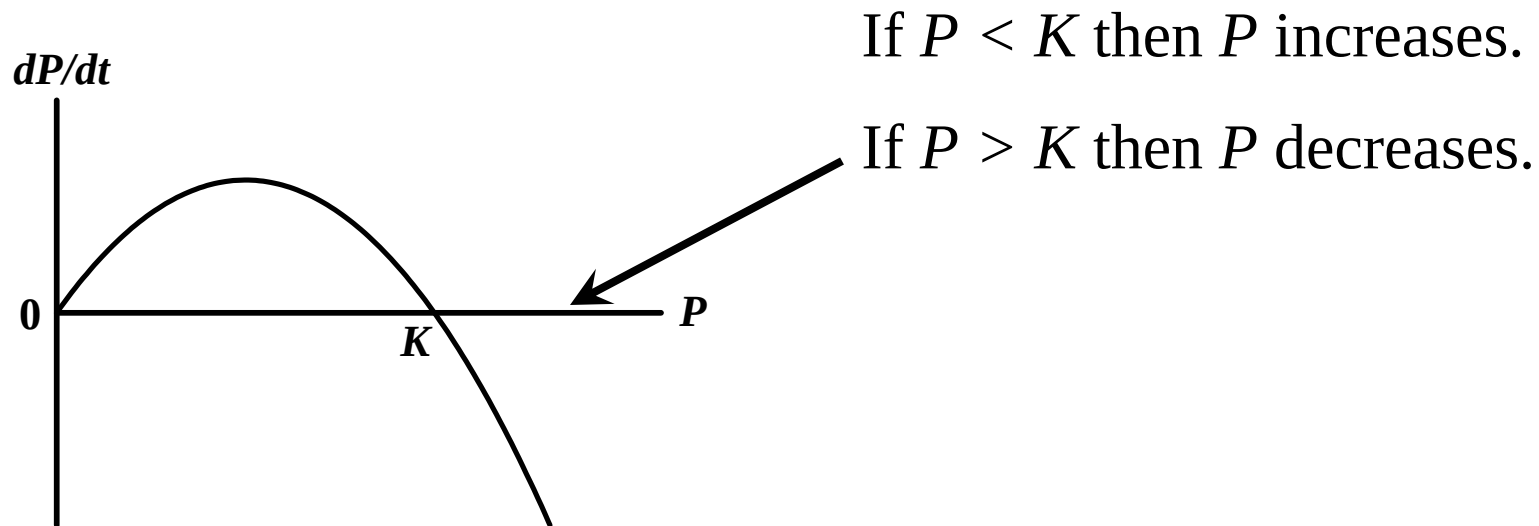
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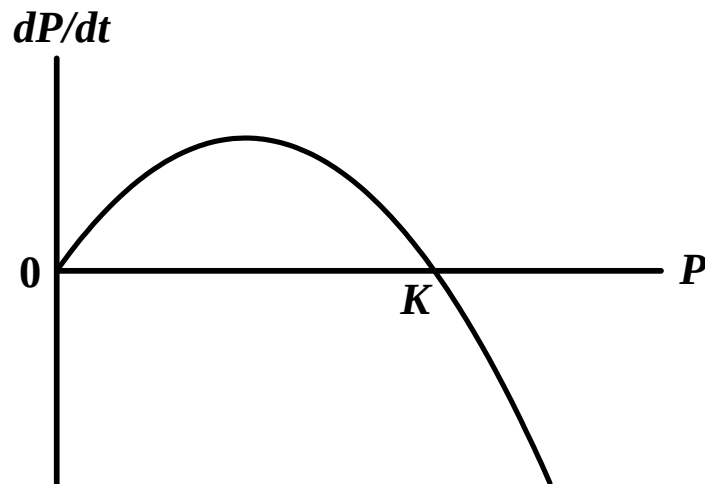
Polynomial Functions

An Application

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If $P < K$ then P increases.

If $P > K$ then P decreases.

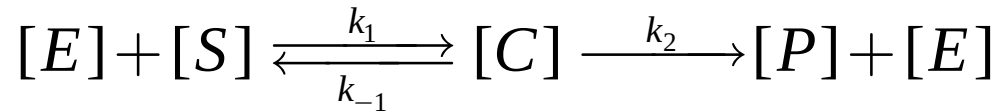
Conclusion

As time increases $P(t)$ approaches K .

(K is called the “carrying capacity”)

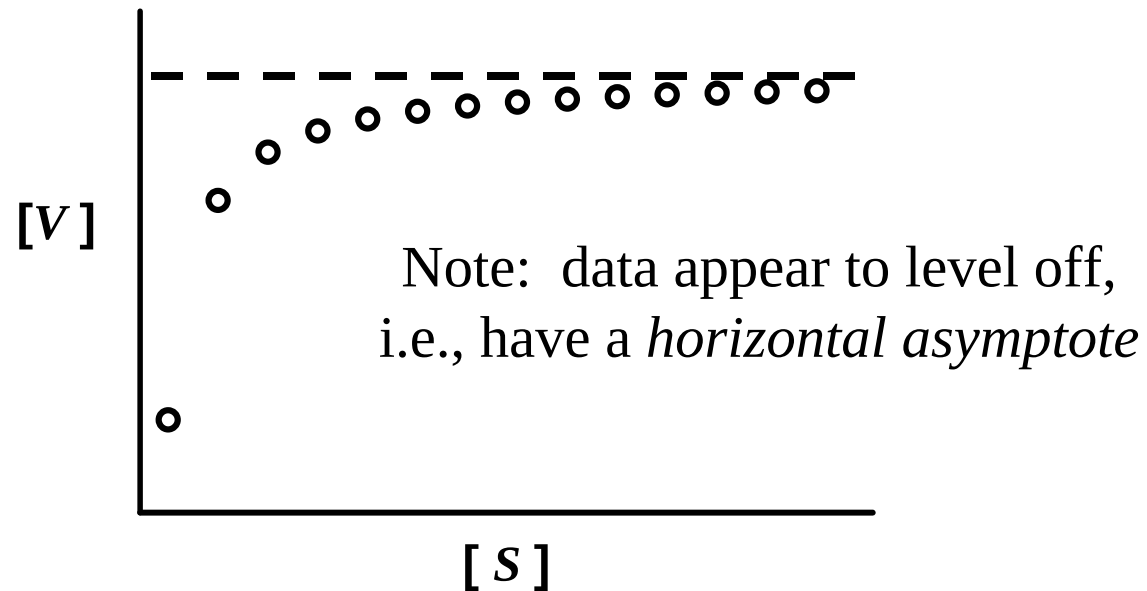
Rational Functions

Enzyme-substrate reactions



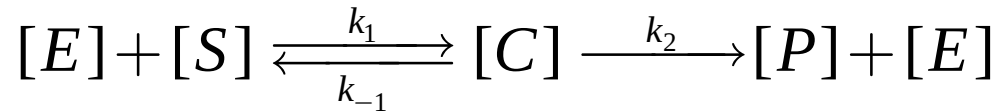
typically produce graphs similar to ...

Reaction Rate vs. Substrate Concentration



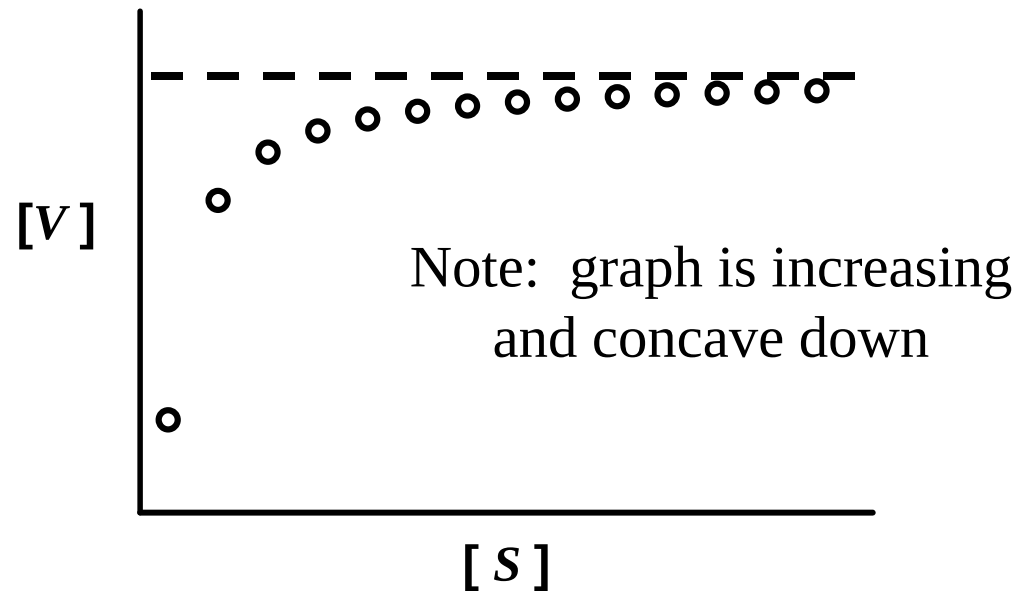
Rational Functions

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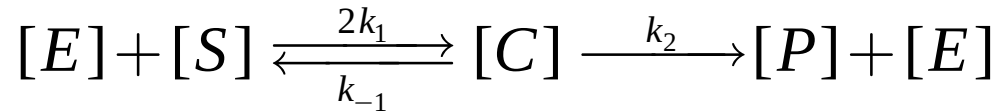
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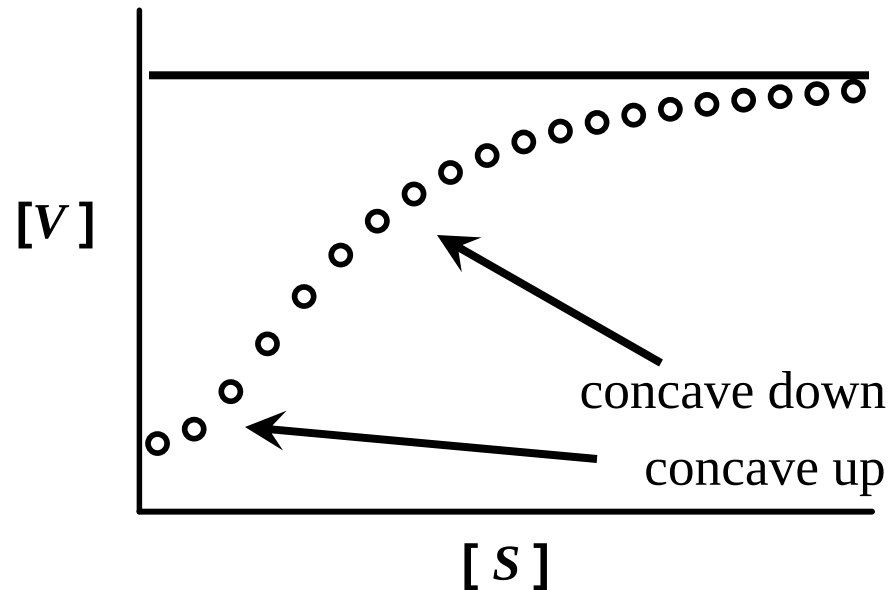
Rational Functions

Cooperative enzyme-substrate reactions



typically produce graphs similar to ...

Reaction Rate vs. Substrate Concentration



Rational Functions

Polynomials, exponentials, logarithms, power functions
do *not* have horizontal asymptotes.

So ... in enzyme-substrate reactions
what is the functional relationship between
the substrate concentration $[S]$ and

the reaction rate $V = \frac{d[S]}{dt}$?

Rational Functions

Next semester, we will show how the *Law of Mass Action* theoretically implies

$$V \approx \frac{a[S]}{b + [S]}$$

$$V \approx \frac{a[S]^2}{b + [S]^2}$$

(cooperative reaction)

These are examples of “rational functions”.